On nonlinear parabolic variational inequalities containing nonlocal terms

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Abstract

We investigate nonlinear parabolic variational inequalities which contain functional dependence on the unknown function, by using the theory of operators of monotone type.

1 Introduction

In this paper we investigate nonlinear parabolic variational inequalities which contain functional dependence on the unknown function. Such parabolic functional differential equations were studied e.g. by L. Simon in [8] (which was motivated by the work of M. Chipot and L. Molinet in [4]), where the following equation was considered:

$$D_t u(t,x) - \sum_{i=1}^{n} D_i[a_i(t,x,u(t,x),Du(t,x);u)] + a_0(t,x,u(t,x),Du(t,x);u) = f(t,x) \tag{1}$$

$$(t,x) \in Q_T = (0,T)\times\Omega, \quad a_i\colon Q_T \times \mathbb{R}^{n+1} \times L^p(0,T;V) \to \mathbb{R},$$

where V denotes a closed linear subspace of the Sobolev-space $W^{1,p}(\Omega)$ $(2 \le p < \infty)$. In the above mentioned paper existence of weak solutions of the above equation is shown. These results were extended to systems of functional differential equations in [2]. In the following, we extend these existence results to variational inequalities by using the (less known) results of [6]. Finally, we show some examples.

2 Parabolic variational inequalities

2.1 Notations

First we introduce some notations. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the C^1 regularity property and $2 \le p < \infty$ be a real number. Denote by $W^{1,p}(\Omega)$ the usual Sobolev-space with the norm

$$\|v\|_{W^{1,p}(\Omega)} = \left(\int_\Omega (|v|^p + \sum_{i=1}^{n} |D_i v|^p)\right)^{\frac{1}{p}},$$

where D_i denotes the distributional derivative with respect to the i-th variable (later we use the notation $D = (D_1,\dots,D_n)$). In addition, let V be a closed linear subspace of the space $W^{1,p}(\Omega)$ which contains $W_0^{1,p}(\Omega)$ (the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$) and let $L^p(0,T;V)$ be the Banach space of measurable functions $u\colon (0,T) \to V$ such that $\|u\|_V^p$ is integrable and the norm is given by

$$\|u\|_{L^p(0,T;V)} = \left(\int_0^T \|u(t)\|_V^p dt\right)^{\frac{1}{p}}.$$

The dual space of $L^p(0,T;V)$ is $L^q(0,T;V^*)$ where $\frac{1}{p} + \frac{1}{q} = 1$ and V^* is the dual space of V. In what follows, we use the notations $X = L^p(0,T;V)$ and $H = L^2(\Omega)$. The pairing between V^* and V is denoted by $\langle\cdot,\cdot\rangle$, further, $[\cdot,\cdot]$ stands for the pairing between X^* and X. In this paper for the derivative of a function $u \in L^p(0,T;V)$ (with respect to the variable t) we use the notation $D_t u$. It is well known (see [9]) that if $D_t u \in X^*$ then $u \in C([0,T],H)$ so that $u(0)$ makes sense.

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2.2 The problem

Let $A\colon X \to X^*$ be a (nonlinear) operator and let $\Phi\colon X \to (-\infty, +\infty]$ be a convex lower semicontinuous function. The effective domain of Φ is the set $D(\Phi) = \{v \in X : \Phi(v) < +\infty\}$. The lower semicontinuity of Φ means that if $u_n \to u$ in X then we have $\liminf_{n\to\infty} \Phi(u_n) \geq \Phi(u)$. We note that if Φ is a convex lower semicontinuous function then Φ is lower semicontinuous with respect to the weak topology of X. An important (later used) property of the above function Φ is that it can be estimated from below relative to an affine function. More precisely, if $u \in D(\Phi)$ then there exists $u^* \in X^*$ and $r \in \mathbb{R}$ such that $\Phi(v) \geq [u^*, v-u] + r$ for all $v \in X$ (see, e.g. [7] or [9]).

Using the above introduced notations a parabolic variational inequality means the following:

(2) to given $u_0 \in H$ and $F \in X^*$ find $u \in D(\Phi)$ such that $D_t u \in X^*$, $u(0) = u_0$ and

$$[D_t u + A(u) - F, v - u] + \Phi(v) - \Phi(u) \geq 0 \quad \text{for all } v \in X. \tag{3}$$

If A is a maximal monotone operator then the existence of solutions of this problem is widely known (see [9]). But in the case of operator containing nonlocal terms the maximal monotonicity can not be guaranteed. In this case we define a weak form of the above problem and prove existence of solutions for this weak form.

In order to define the weak form observe that if $u, v \in X$ are such that $D_t u, D_t v \in X^*$ then we have

$$\begin{aligned}(4)\quad [D_t u, v-u] &= [D_t u - D_t v, v - u] + [D_t v, v-u] = \\ &= \frac{1}{2}\left(\|v(0)-u(0)\|_H^2 - \|v(T)-u(T)\|_H^2\right) + [D_t v, v-u] \leq \frac{1}{2}\|v(0)-u(0)\|_H^2 + [D_t v, v-u].\end{aligned}$$

In the above transformation we used integration by parts and Newton-Leibniz's rule (in X, X^*). As a consequence it is natural to define the weak form of (2)–(3) by the following:

(5) to given $u_0 \in H$ and $F \in X^*$ find $u \in D(\Phi)$ such that there exists a Banach space B such that

(6) $H \subset B$, the embedding $H \hookrightarrow B$ is continuous, $u \in C([0,T], B)$ and $u(0) = u_0$, further,

(7) $[D_t v + A(u) - F, v - u] + \Phi(v) - \Phi(u) \geq -\frac{1}{2}\|v(0) - u_0\|_H^2$ for all $v \in X$ such that $D_t v \in X^*$.

It is obvious that if u is a solution of (2)–(3) then u is the solution of (5)–(7) with $B = H$. The next statement shows the converse: if u is a solution of (5)–(7) such that $D_t u \in X^*$ then u satisfies (2)–(3), too (see [6]).

Proposition 1. *Let $A\colon X \to X^*$ be an operator and $\Phi\colon X \to (-\infty, +\infty]$ a convex lower semicontinuous function. If* (5)–(7) *has a solution $u \in D(\Phi)$ such that $D_t u \in X^*$ then u is a solution of the problem* (2)–(3)*, too.*

Proof. Let u be a solution of (5)–(7) such that $D_t u \in X^*$. Since $D_t u \in X^*$ and $H \hookrightarrow B$ is continuous, $u(0) = u_0$ holds in the usual sense (i.e., in the sense of $u \in C([0,T], H)$). Now let $w \in X$ be an arbitrary element such that $D_t w \in X^*$ and $w(0) = u_0$. Then by choosing $v = (1-\lambda)u + \lambda w$ in (7) and using $v(0) = u_0$, the convexity of Φ we obtain

$$\lambda[(1-\lambda)D_t u + A(u) - F, w - u] + \lambda^2 [D_t w, w - u] + \lambda\left(\Phi(w) - \Phi(u)\right) \geq 0.$$

Dividing by λ we get from this as $\lambda \to 0$ that

$$[D_t u + A(u) - F, w - u] + \Phi(w) - \Phi(u) \geq 0 \tag{8}$$

for all $w \in X$ such that $D_t w \in X^*$ and $w(0) = u_0$. Now let $v \in X$ be an arbitrary element and choose $w_\varepsilon \in X$ elements such that $D_t w_\varepsilon \in X^*$, $w_\varepsilon(0) = u_0$ and $w_\varepsilon \to v$ in X as $\varepsilon \to 0$; e.g. the following functions are suitable:

$$w_\varepsilon(t) = e^{-\frac{t}{\varepsilon}} u_0 + \frac{1}{\varepsilon}\int_0^t e^{\frac{s-t}{\varepsilon}} v(s)ds.$$

By substituting $w = w_\varepsilon$ in inequality (8) and by using the convergence of the sequence (w_ε) and the convex lower semicontinuity of Φ we obtain as $\varepsilon \to 0$ that

$$[D_t u + A(u) - F, v - u] + \Phi(v) - \Phi(u) \geq 0$$

for all $v \in X$. This means that u is a solution of (2)–(3). $\square$

In the following we consider only special functions Φ. Namely, let $\varphi\colon V \to (-\infty, +\infty]$ be a convex lower semicontinuous function and define the $\Phi\colon X \to (-\infty, +\infty]$ function by the following:

$$\Phi(u) = \begin{cases} \displaystyle\int_0^T \varphi(u(t))dt, & \text{if } \varphi(u(\cdot)) \in L^1(0,T) \\ +\infty, & \text{otherwise.} \end{cases} \tag{9}$$

Since φ can be estimated from below relative to an affine function, so that Φ maps actually to $(-\infty, +\infty]$. It is easy to verify that Φ is a convex lower semicontinuous function (see [9]). In what follows, we study only such functions Φ.

2.3 Existence of solutions

The next basic result shows the existence of solutions of problem (5)–(7) in case of a not necessarily monotone operator A. This theorem is stated and proved in [6] in the case of $p = 2$ and of the family of operators $A(t)\colon V \to V^*$ ($t \in (0,T)$). For the convenience, we formulate the conditions before stating the theorem.

V1. Suppose that $A\colon X \to X^*$ is a bounded, coercive operator which satisfies the following $(\star)$ condition:

   $(\star)$ for all $(v_n) \subset X$ sequence such that $v_n \to v$ weakly in X, $v_n \to v$ strongly in $L^p(0,T; L^p(\Omega))$ and $\limsup_{n\to\infty}[A(v_n), v_n - v] \le 0$, follows that $[A(v), v - w] \le \liminf_{n\to\infty}[A(v_n), v_n - w]$ for all $w \in X$.

V2. Suppose that $\varphi\colon V \to (-\infty, +\infty]$ is a proper convex lower semicontinuous function (by proper we mean that $\varphi \not\equiv +\infty$) such that $(0,0) \in \partial\varphi$ (where $\partial\varphi$ is the subdifferential of φ, see e.g. [9]).

V3. Suppose that there exists a closed subspace $V_0 \subset V$ such that V_0 is dense in H and the embedding $V_0 \hookrightarrow H$ is continuous, further, $\varphi(v + v_0) = \varphi(v)$ for all $v \in V$ and all $v_0 \in V_0$.

Theorem 2. *Suppose that conditions V1–V3 are satisfied. Then for all $F \in X^*$ there exists $u \in D(\Phi)$ such that $u \in C([0,T], V_0^*)$, $D_t u \in L^q(0,T; V_0^*)$, $u(0) = 0$ and*

$$[D_t v + A(u) - F, v - u] + \Phi(v) - \Phi(u) \ge -\frac{1}{2}\|v(0)\|_H^2 \quad \textit{for all } v \in X \textit{ such that } D_t v \in X^*.$$

Proof. The proof is based on the proof of Theorem 3.3 found in [6].

Step 1 Let $J\colon V \to V^*$ be the duality map of the space V which is (since V and V^* are strictly convex spaces) singlevalued, bounded, strictly monotone, demicontinuous, pseudomonotone and coercive. Since $\partial\varphi\colon V \to 2^{V^*}$ is a maximal monotone operator and J has the preceding properties thus for all $v \in V$ and for all $\varepsilon > 0$ the problem

$$J(v_\varepsilon - v) + \varepsilon\partial\varphi(v_\varepsilon) \ni 0$$

has got a uniqe $v_\varepsilon \in D(\varphi)$ solution (see [9]). Then we can define operators $F_\varepsilon\colon V \to V$ and $B_\varepsilon\colon V \to V^*$ given by the following formulas:

$$F_\varepsilon(v) = v_\varepsilon, \qquad B_\varepsilon(v) = -\frac{1}{\varepsilon}J(v_\varepsilon - v).$$

The following lemma shows that the above defined operators have important properties. For the proof see section III.3.1 in [7].

Lemma 3. *The above defined operator $F_\varepsilon\colon V \to V$ is bounded, maps in $D(\varphi)$ and $F_\varepsilon(0) = 0$, further, operator $B_\varepsilon\colon V \to V^*$ is bounded (moreover $\|B_\varepsilon(v)\|_{V^*} \le \frac{1}{\varepsilon} \cdot \|v\|_V$), monotone, demicontinuous (hence maximal monotone) and $B_\varepsilon(0) = 0$.*

Now consider the function $\varphi_\varepsilon\colon V \to [-\infty, +\infty]$ given by

$$\varphi_\varepsilon(v) = \inf_{w\in V}\left\{\frac{1}{2\varepsilon}\|w - v\|_V^2 + \varphi(w)\right\}.$$

It is crucial in the proof of our theorem that this function φ_ε is closely related to the above operators F_ε, B_ε. This relation is shown by the following lemma. The proof can be found in section III.3.2 in [7].

Lemma 4. *The above function φ_ε maps in $(-\infty,+\infty)$ (in other words it is everywhere finite), $\varphi_\varepsilon \le \varphi$, further,*

$$\varphi_\varepsilon(v) = \frac{1}{2\varepsilon}\|F_\varepsilon(v) - v\|_V^2 + \varphi(F_\varepsilon(v)), \qquad \varphi_\varepsilon(w) \ge \varphi_\varepsilon(v) + \langle B_\varepsilon(v), w - v\rangle \tag{10}$$

for all $v, w \in V$, where operators F_ε, B_ε are defined above.

We note that the proofs of the above lemmas use from the properties of φ only that it is a proper lower semicontinuous function on V such that $(0,0) \in \partial\varphi$. Observe that $(0,0) \in \partial\varphi$ means $\varphi(v) \ge \varphi(0)$. Till now we have not used assumption V3, however it plays a crucial role in the proof of Theorem 2. First, observe that from the definition of v_ε and B_ε it follows that $B_\varepsilon(v) \in \partial\varphi(v_\varepsilon)$ for all $v \in V$. Therefore, by using the definition of the subdifferential and condition V3 we obtain that for all $v \in V$ and $v_0 \in V_0$

$$\langle B_\varepsilon(v), v_0\rangle = \langle B_\varepsilon(v), (v_\varepsilon + v_0) - v_\varepsilon\rangle \le \varphi(v_\varepsilon + v_0) - \varphi(v_\varepsilon) = 0. \tag{11}$$

By choosing $-v_0$ instead of v_0 in the above inequality, we get $\langle B_\varepsilon(v), v_0\rangle = 0$ for all $v_0 \in V_0$ and $\varepsilon > 0$.

Now let us introduce operator $\mathcal{B}_\varepsilon \colon X \to X^*$ by $[\mathcal{B}_\varepsilon(u)](t) = B_\varepsilon(u(t))$. Then from the properties of operator B_ε it is easy to see that $\mathcal{B}_\varepsilon$ is bounded, monotone, demicontinuous, thus maximal monotone, further, $\mathcal{B}_\varepsilon(0) = 0$ (so $[\mathcal{B}_\varepsilon(u), u] \ge 0$ for all $u \in X$). In addition, from (10) it follows that

$$\Phi_\varepsilon(v) = \frac{1}{2\varepsilon}\int_0^T \|F_\varepsilon(v(t)) - v(t)\|_V^2 dt + \Phi(F_\varepsilon(v)), \qquad \Phi_\varepsilon(w) \ge \Phi_\varepsilon(v) + [\mathcal{B}_\varepsilon(v), w - v] \tag{12}$$

for all $v, w \in X$ and all $\varepsilon > 0$, where

$$\Phi_\varepsilon(u) = \begin{cases} \int_0^T \varphi_\varepsilon(u(t))dt, & \text{if } \varphi_\varepsilon(u(\cdot)) \in L^1(0,T) \\ +\infty, & \text{otherwise.} \end{cases} \tag{13}$$

Further, from (11) we obtain

$$[\mathcal{B}_\varepsilon(u), u_0] = 0 \text{ for all } u \in X, u_0 \in L^p(0,T;V_0) \text{ and } \varepsilon > 0. \tag{14}$$

Step 2 Let us define operator $L : D(L) \to X^*$ by the formula

$$D(L) = \{u \in X \colon D_t u \in X^*, u(0) = 0\}, \qquad Lu = D_t u.$$

It is well known that L is a maximal monotone operator (see [9]). Since $L\colon D(L) \to X^*$ and $\mathcal{B}_\varepsilon \colon X \to X^*$ are maximal monotone operators, their sum is also maximal monotone. Then it follows from the properties of A that for all $\varepsilon > 0$ there exists $u_\varepsilon \in X$ such that $D_t u_\varepsilon \in X^*$,

$$D_t u_\varepsilon + A(u_\varepsilon) + \mathcal{B}_\varepsilon(u_\varepsilon) = F, \tag{15}$$

and $u_\varepsilon(0) = 0$ (see [1] or [5]). By applying both sides of the above equation to u_ε and using Newton-Leibniz's rule and the monotonicity of $\mathcal{B}_\varepsilon$ we obtain

$$\frac{[A(u_\varepsilon), u_\varepsilon]}{\|u_\varepsilon\|_X} \le \|F\|_{X^*}.$$

Since A is coercive, $\|u_\varepsilon\|_X$ is bounded. Then due to the boundedness of A, $(A(u_\varepsilon))$ is also bounded. In addition, by applying both sides of equation (15) to an arbitrary element of $L^p(0,T;V_0)$ and using (14) we obtain the boundedness of $(\|D_t u_\varepsilon\|_{L^q(0,T;V_0^*)})$. Moreover, if we apply both sides of (15) to $\chi_{(0,t)}u_0$ where $\chi_{(0,t)}$ is the characteristic function of the interval $(0,t)$ and $u_0 \in L^p(0,T;V_0)$ we obtain that $\|u_\varepsilon(t)\|_H$ is also bounded for all $t \in [0,T]$.

Step 3 It follows from the preceding step that there exists a subsequence of (u_ε) (which is denoted again by (u_ε) for simplicity) and $u \in X$ such that as $\varepsilon \to 0$

$$u_\varepsilon \to u \text{ weakly in } X, \tag{16}$$
$$u_\varepsilon \to u \text{ strongly in } L^p(0,T;L^p(\Omega)), \tag{17}$$
$$D_t u_\varepsilon \to D_t u \text{ weakly in } L^q(0,T;V_0^*), \tag{18}$$
$$A(u_\varepsilon) \to w^* \text{ weakly in } L^q(0,T;V^*), \tag{19}$$
$$u_\varepsilon(t) \to z_t \text{ weakly in } H \text{ for all } t \in [0,T]. \tag{20}$$

To obtain (17) one uses the compactness of the embedding $V_0 \hookrightarrow L^p(\Omega)$ ($V \hookrightarrow L^p(\Omega)$ is a compact embedding), then we can apply the well known embedding theorem (see [5]).

We show that if $u \in X$ and $D_t u \in L^q(0,T;V_0^*)$ then $u \in C([0,T],V_0^*)$. To see this, set

$$v(t) = \int_0^t D_t u(\tau) d\tau.$$

Then $v \in C([0,T],V_0^*)$ since

$$\|v(t) - v(s)\|_{V_0^*} \leq \int_s^t \|D_t u(\tau)\|_{V_0^*} d\tau$$

and the integral is absolute continuous. In addition, $D_t v = D_t u$ hence $v(t) = u(t) + c$ for a.a. $t \in [0,T]$ where $c \in V_0^*$.

Now we prove that $u(0) = 0$. To this end, observe that for all $v_0 \in V_0$

$$\langle u_\varepsilon(t), v_0 \rangle_{V_0} = \int_0^t \langle D_t u_\varepsilon(\tau), v_0 \rangle_{V_0} d\tau.$$

Then convergences (18) and (20) imply that

$$\langle z_t, v_0 \rangle_{V_0} = \int_0^t \langle D_t u(\tau), v_0 \rangle_{V_0} d\tau$$

for all $v_0 \in V_0$. From this it follows that $z_t = u(t) - u(0)$ which means $u_\varepsilon(t) \to u(t) - u(0)$ weakly in V_0^* for all $t \in [0,T]$. Convergence (17) implies that for a subsequence $u_\varepsilon(t) \to u(t)$ strongly in $L^p(\Omega)$ for a.a. $t \in [0,T]$. Hence from these preceding convergences it follows that $u(t) = u(t) - u(0)$ for a.a. $t \in [0,T]$ thus $u(0) = 0$.

Step 4 Now let $v \in X$ be such that $D_t v \in L^q(0,T;V^*)$. Then by applying both sides of (15) to $v - u_\varepsilon$ and using the method of passing to weak form (see (4)) we obtain that

$$[D_t v + A(u_\varepsilon) + \mathcal{B}_\varepsilon - F, v - u_\varepsilon] + \frac{1}{2}\|v(0)\|_H^2 \geq 0.$$

By using (12) we have

$$[D_t v + A(u_\varepsilon) - F, v - u_\varepsilon] + \Phi_\varepsilon(v) + \frac{1}{2}\|v(0)\|_H^2 \geq \frac{1}{2\varepsilon}\int_0^T \|F_\varepsilon(u_\varepsilon(t)) - u_\varepsilon(t)\|_V^2 dt + \Phi(F_\varepsilon(u_\varepsilon)).$$

Observe that $\Phi_\varepsilon \leq \Phi$ thus from the above inequality it follows that

$$[D_t v + A(u_\varepsilon) - F, v - u_\varepsilon] + \Phi(v) + \frac{1}{2}\|v(0)\|_H^2 \geq \frac{1}{2\varepsilon}\int_0^T \|F_\varepsilon(u_\varepsilon(t)) - u_\varepsilon(t)\|_V^2 dt + \Phi(F_\varepsilon(u_\varepsilon)). \tag{21}$$

for all $v \in X$ such that $D_t v \in X^*$. By choosing $v = 0$ ($0 \in D(\Phi)$ due to condition V2) in the above inequality, it follows that $\Phi(F_\varepsilon(u_\varepsilon)) < +\infty$. On the other hand, $\Phi(F_\varepsilon(u_\varepsilon)) \geq \Phi(0)$, therefore $\int_0^T \|F_\varepsilon(u_\varepsilon(t)) - u_\varepsilon(t)\|_V^2 dt \xrightarrow{\varepsilon \to 0} 0$, hence by using (16) we have that $F_\varepsilon(u_\varepsilon(\cdot)) \xrightarrow{\varepsilon \to 0} u$ weakly in X. Then by using this and the lower semicontinuity of Φ with respect to the weak convergence we obtain that $\liminf_{\varepsilon \to 0} \Phi(F_\varepsilon(u_\varepsilon)) \geq \Phi(u)$. By applying the above convergences it follows from (21) as $\varepsilon \to 0$ that $\Phi(u) < +\infty$ thus $u \in D(\Phi)$. From (21) we have that

$$[D_t v - F, v - u_\varepsilon] + [A(u_\varepsilon), v - u] + \Phi(v) - \Phi(F_\varepsilon(u_\varepsilon)) + \frac{1}{2}\|v(0)\|_H^2 \geq [A(u_\varepsilon), u_\varepsilon - u], \tag{22}$$

thus from (16) and (19) it follows that

$$\limsup_{\varepsilon \to 0} [A(u_\varepsilon), u_\varepsilon - u] \leq [D_t v + w^* - F, v - u] + \Phi(v) - \Phi(u) + \frac{1}{2}\|v(0)\|_H^2 \tag{23}$$

for all $v \in X$ such that $D_t v \in X^*$.

Step 5 We show that for all $\delta > 0$ there exists $v \in X$ such that $D_t v \in X^*$ and the right hand side of the above inequality (23) is less than δ. Let $\delta > 0$ and suppose that

$$[D_t v + w^* - F, v - u] + \Phi(v) - \Phi(u) + \frac{1}{2}\|v(0)\|_H^2 \geq \delta > 0 \tag{24}$$

for all $v \in X$ such that $D_t v \in X^*$. Let

$$\tilde{u}_\varepsilon(t) = \frac{1}{\varepsilon}\int_0^t e^{\frac{s-t}{\varepsilon}} u(s)ds.$$

Then $\tilde{u}_\varepsilon \in D(\Phi)$ (because $u \in D(\Phi)$), $\tilde{u}_\varepsilon \to u$ in X as $\varepsilon \to 0$, further, $\tilde{u}_\varepsilon(t) + \varepsilon D_t \tilde{u}_\varepsilon(t) = u(t)$ for a.a. $t \in (0,T)$ and $\tilde{u}_\varepsilon(0) = 0$. From these it follows that $D_t \tilde{u}_\varepsilon \in X \subset X^*$ $(p \geq 2)$. Since φ is a convex lower semicontinuous function therefore it can be estimated from below relative to an affine function. Thus, there exists $g^* \in V^*$ and $r_0 \in \mathbb{R}$ such that the function $\tilde{\varphi} \colon V \to (-\infty, +\infty]$ given by $\tilde{\varphi}(v) := \varphi(v) - \langle g^*, v\rangle - r_0$ is nonnegative, convex lower semicontinuous. Then from Jensen's inequality we obtain that for a.a. $t \in (0,T)$

$$\tilde{\varphi}(\tilde{u}_\varepsilon(t)) = \tilde{\varphi}\left(\frac{1}{\varepsilon}\int_0^t e^{\frac{s-t}{\varepsilon}} u(s)ds\right) = \tilde{\varphi}\left(\int_{e^{-\frac{t}{\varepsilon}}}^1 u(t+\varepsilon\log r)dr\right) \leq$$

$$\leq \int_{e^{-\frac{t}{\varepsilon}}}^1 \tilde{\varphi}(u(t+\varepsilon\log r))dr = \frac{1}{\varepsilon}\int_0^t e^{\frac{s-t}{\varepsilon}} \tilde{\varphi}(u(s))ds,$$

where $r = e^{\frac{s-t}{\varepsilon}}$. By integrating both sides of the above inequality in $(0,T)$ it follows that

$$\int_0^T \tilde{\varphi}(\tilde{u}_\varepsilon(t))dt \leq \int_0^T \int_0^t \frac{1}{\varepsilon} e^{\frac{s-t}{\varepsilon}} \tilde{\varphi}(u(s))ds\,dt = \int_0^T \int_s^T \frac{1}{\varepsilon} e^{\frac{s-t}{\varepsilon}} \tilde{\varphi}(u(s))dt\,ds =$$

$$= \int_0^T \left(1 - e^{s-T}\right) \tilde{\varphi}(u(s))ds \leq \int_0^T \tilde{\varphi}(u(s))ds.$$

By using the definition of $\tilde{\varphi}$ we obtain

$$\Phi(\tilde{u}_\varepsilon) - \Phi(u) \leq \int_0^T \langle g^*, \tilde{u}_\varepsilon(t) - u(t)\rangle\, dt. \tag{25}$$

Now let us substitute $v = \tilde{u}_\varepsilon \in D(\Phi)$ in (24) and observe that $v_\varepsilon(0) = 0$ and $[D_t\tilde{u}_\varepsilon, \tilde{u}_\varepsilon - u] = \frac{1}{\varepsilon}[u - \tilde{u}_\varepsilon, \tilde{u}_\varepsilon - u] \leq 0$. Thus by using (25) from (24) we obtain that

$$0 < \delta \leq \int_0^T \langle w^*(t) - F(t) + g^*, \tilde{u}_\varepsilon(t) - u(t)\rangle\, dt.$$

This is a contradiction since $\tilde{u}_\varepsilon \to u$ weakly in X as $\varepsilon \to 0$ so that the right hand side of the above inequality tends to 0 as $\varepsilon \to 0$.

Step 6 From the preceding step we obtain $\limsup\limits_{\varepsilon\to 0}[A(u_\varepsilon), u_\varepsilon - u] \leq 0$ thus from the property $(\star)$ it follows that

$$[A(u), u - v] \leq \liminf_{\varepsilon\to 0}[A(u_\varepsilon), u_\varepsilon - v]$$

for all $v \in X$. By using this relation in (22) we obtain as $\varepsilon \to 0$ that

$$[D_t v + A(u) - F, v - u] + \Phi(v) - \Phi(u) \geq -\frac{1}{2}\|v(0)\|_H^2$$

for all $v \in X$ such that $D_t v \in X^*$. The proof of Theorem 2 is complete. □

Remark 5. It is obvious that in Theorem 2 with $V_0 = V$ the stronger assertion $u \in C([0,T], H)$ holds.

Proposition 6. *Suppose that the conditions of Theorem 2 are satisfied, further, operator A is strictly monotone. Then the solution u included in Theorem 2 is unique.*

Proof. We use arguments of [6]. Let u_1 and u_2 be elements of $D(\Phi)$ such that $u_i \in C([0,T], V_0^*)$, $D_t u_i \in L^q(0,T;V_0^*)$, $u_i(0) = 0$ and

$$[D_t v + A(u_i) - F, v - u_i] + \Phi(v) - \Phi(u_i) \geq -\frac{1}{2}\|v(0)\|_H^2 \quad \text{for all } v \in X \text{ such that } D_t v \in X^*.$$

By adding the above inequality for $i = 1, 2$ it follows that

$$[D_t v - F, 2v - (u_1 + u_2)] + [A(u_1), v - u_1] + [A(u_2), v - u_2] + 2\Phi(v) - \Phi(u_1) - \Phi(u_2) \geq -\|v(0)\|_H^2.$$

Set $\tilde{u} = \frac{u_1+u_2}{2}$. Then by dividing by 2 and using the convexitiy of Φ after elementary transformations we obtain that

$$(26) \quad [D_t v + \frac{1}{2}(A(u_1) + A(u_2)) - F, v - \tilde{u}] + \Phi(v) - \Phi(\tilde{u}) + \frac{1}{2}\|v(0)\|_H^2 \geq \frac{1}{4}[A(u_1) - A(u_2), u_1 - u_2].$$

for all $v \in X$ such that $D_t v \in X^*$. By using arguments of Step 5 in the proof of Theorem 2 with $w^* = \frac{1}{2}(A(u_1) + A(u_2))$ and $u = \tilde{u}$ it follows that for arbitrarily chosen $\delta > 0$ there exists $v \in X$ such that $D_t v \in X^*$ and the left hand side of the inequality (26) is less than δ. This yields

$$0 \geq [A(u_1) - A(u_2), u_1 - u_2].$$

Since A is a strictly monotone operator, it follows that $u_1 = u_2$. □

3 Nonlocal variational inequalities

In the following we apply Theorem 2 to operator A which can be obtained from the elliptic part of equation (1). Let $a_i \colon Q_T \times \mathbb{R}^{n+1} \times L^p(0,T;V) \to \mathbb{R}$ $(i = 0, \dots, n)$ and define operator $A \colon X \to X^*$ with the following formula: for $u, v \in X$ let

$$(27) \;\; [A(u), v] := \int_{Q_T} \left[\sum_{i=1}^{n} a_i(t,x,u(t,x),Du(t,x);u) D_i v(t,x) + a_0(t,x,u(t,x),Du(t,x);u) v(t,x) \right] dtdx.$$

Now we formulate 5 assumptions which will imply that A satisfies conditon V1. To formulate these conditions we introduce a notation: a vector $\xi \in \mathbb{R}^{n+1}$ is written in the form $\xi = (\zeta_0, \zeta)$ where $\zeta_0 \in \mathbb{R}$ and $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{R}^n$.

F1. Suppose that $a_i \colon Q_T \times \mathbb{R}^{n+1} \times L^p(0,T;V) \to \mathbb{R}$ are Carathéodory functions for each fixed $v \in L^p(0,T;V)$. This means that they are measurable in (t,x) for every $(\zeta_0, \zeta) \in \mathbb{R}^{n+1}$, and continuous in (ζ_0, ζ) for a.a. $(t,x) \in Q_T$ $(i = 0, \dots, n)$.

F2. Suppose that there exist bounded operators $g_1 \colon L^p(0,T;V) \to \mathbb{R}^+$ and $k_1 \colon L^p(0,T;V) \to L^q(Q_T)$ such that

$$|a_i(t,x,\zeta_0,\zeta;v)| \leq g_1(v) \left(|\zeta_0|^{p-1} + |\zeta|^{p-1}\right) + [k_1(v)](t,x)$$

for a.a. $(t,x) \in Q_T$, each $(\zeta_0, \zeta) \in \mathbb{R}^{n+1}$ and $v \in L^p(0,T;V)$ $(i = 0, \dots, n)$.

F3. Suppose that for each $\zeta \neq \eta \in \mathbb{R}^n$, a.a. $(t,x) \in Q_T$, each $\zeta_0 \in \mathbb{R}$ and $v \in L^p(0,T;V)$

$$\sum_{i=1}^{n} \left(a_i(t,x,\zeta_0,\zeta;v) - a_i(t,x,\zeta_0,\eta;v)\right)(\zeta_i - \eta_i) > 0.$$

F4. Suppose that there exist operartors $g_2 \colon L^p(0,T;V) \to \mathbb{R}^+$ and $k_2 \colon L^p(0,T;V) \to L^1(Q_T)$ such that

$$\sum_{i=0}^{n} a_i(t,x,\zeta_0,\zeta;v)\zeta_i \geq g_2(v)\left(|\zeta_0|^p + |\zeta|^p\right) - [k_2(v)](t,x)$$

for a.a. $(t,x) \in Q_T$, each $(\zeta_0, \zeta) \in \mathbb{R}^{n+1}$ and $v \in L^p(0,T;V)$. Further, operators g_2, k_2 has the following property:

$$\lim_{\|v\|_{L^p(0,T;V)} \to \infty} \left(g_2(v)\|v\|_{L^p(0,T;V)}^{p-1} - \frac{\|k_2(v)\|_{L^1(Q_T)}}{\|v\|_{L^p(0,T;V)}} \right) = +\infty.$$

F5. Suppose that if $u_k \to u$ weakly in $L^p(0,T;V)$ and strongly in $L^p(0,T;L^p(\Omega))$, then for $i = 0, \dots, n$

$$\lim_{k\to\infty} \|a_i(\cdot\,, u_k(\cdot), Du_k(\cdot); u_k) - a_i(\cdot\,, u_k(\cdot), Du_k(\cdot); u)\|_{L^q(Q_T)} = 0.$$

Theorem 7. *Assume that conditions F1–F5 are satisfied. Then operator $A \colon X \to X^*$ defined by (27) is bounded, coercive and it has the property $(\star)$.*

Proof. One can easily verify the boundedness of A by using condition F2 and Hölder's inequality. The coerciveness is a trivial consequence of conditon F4 (for the detailed proof see [2, 8]).

The property $(\star)$ can be shown by a minor modification of the proofs found in [2, 8, 3]. We sketch this modification. Let $u_0 \in X$ be an arbitrary fixed element. Then let us define operator $\tilde{A}_{u_0} \colon X \to X^*$ with the following formula:

$$[\tilde{A}_{u_0}(u), v] = \int_{Q_T} \left[\sum_{i=1}^n a_i(t,x,u(t,x),Du(t,x);u_0) D_i v(t,x) + a_0(t,x,u(t,x),Du(t,x);u_0) v(t,x) \right] dt\, dx.$$

We show that operator $\tilde{A}_{u_0}$ has the property $(\star)$. To this end, let $(v_n) \subset X$ be a sequence such that $v_n \to v$ weakly in X and strongly in $L^p(0,T;L^p(\Omega))$, further, $\limsup_{n\to\infty} [\tilde{A}_{u_0}(v_n), v_n - v] \le 0$. Observe that then there exists a subsequence $(\tilde{v}_n) \subset (v_n)$ such that $\tilde{v}_n \to v$ almost everywhere in Q_T. But then by using the proof found in [3] word for word we obtain that $\tilde{A}_{u_0}(\tilde{v}_n) \to \tilde{A}_{u_0}(v)$ weakly in X^* and $\lim_{n\to\infty} [\tilde{A}_{u_0}(\tilde{v}_n), \tilde{v}_n - v] = 0$ for a suitable subsequence $(\tilde{v}_n)$. From this it follows that $[\tilde{A}_{u_0}(\tilde{v}_n), \tilde{v}_n - w] = [\tilde{A}_{u_0}(\tilde{v}_n), \tilde{v}_n - v] + [\tilde{A}_{u_0}(\tilde{v}_n), v - w] \to [\tilde{A}_{u_0}(v), v - w]$ for all $w \in X^*$. Since every sequence (v_n) has a subsequence for which the implication holds hence it is easy to see that $\tilde{A}_{u_0}$ has actually the property $(\star)$. The proof of the property $(\star)$ for operator A is a trivial modification of the proof found in [2, 8]: one do not need the embedding theorem. □

Corollary 8. *Assume that conditions V2–V3, F1–F5 are fulfilled. Then for every $F \in X^*$ there exists $u \in D(\Phi)$ such that $u \in C([0,T],H)$, $D_t u \in L^q(0,T;V_0^*)$, $u(0) = 0$ and*

$$[D_t v + A(u) - F, v - u] + \Phi(v) - \Phi(u) \ge -\frac{1}{2}\|v(0)\|_H^2 \quad \text{for all } v \in X \text{ such that } D_t v \in X^*,$$

where Φ is given by (9) *and A is given by* (27).

4 Examples

In this section we show some examples for function φ and operator A involved in Theorem 7.

4.1 Function φ

The simpliest example is $\varphi \equiv 0$ (we can with it check if Theorem 7 is usable at all). This function obviously satisfies condition V2 and V3 with $V_0 = V$. Then according to Theorem 7 there is a solution u of (5)–(7) such that $D_t u \in L^q(0,T;V^*)$. Thus from Proposition 1 it follows that u is a solution of problem (2)–(3), too. In this case inequality (3) means that

$$[Lu + A(u) - F, v - u] \ge 0 \quad \text{for all } v \in X. \tag{28}$$

Here, by choosing $v = 0$ and $v = 2u$ we obtain that

$$[Lu + A(u) - F, -u] \ge 0 \quad \text{and} \quad [Lu + A(u) - F, u] \ge 0$$

which means that $[Lu + A(u) - F, u] = 0$. By substituting this result in (28) we get that

$$[Lu + A(u) - F, v] \ge 0 \quad \text{for all } v \in X.$$

By choosing in the above inequality $(-v)$ instead of v the following holds:

$$[Lu + A(u) - F, v] = 0 \quad \text{for all } v \in X.$$

The above argument shows that in this case of φ the variational inequality is the same as the classical variational problem: $Lu + A(u) + F = 0$. Let us denote by N the following second order differential operator: $u \mapsto \sum_{i=1}^n D_i[a_i(t,x,u(t,x),Du(t,x);u)] - a_0(t,x,u(t,x),Du(t,x);u)$, further, let $[F,v] = \int_{Q_T} fv$ where $f \in L^q(Q_T)$. Then it is well known that in case $V = W^{1,p}(\Omega)$ problem $Lu + A(u) + F = 0$ might be considered as the weak form of the following parabolic mixed problem:

$$\begin{gathered} D_t u - N(u) - f = 0 \quad \text{in } Q_T, \\ \partial_{\nu^*} u = 0 \text{ in } (0,T) \times \partial\Omega \text{ and } u = u_0 \text{ in } \{0\} \times \Omega, \end{gathered} \tag{29}$$

where $\partial_{\nu^*}(t,x) = \sum_{i=1}^n a_i(t,x,u(t,x),Du(t,x);u)\nu_i(t,x)$ (ν is the outer unit normal along $(0,T)\times\partial\Omega$).

Now let $V = W^{1,p}(\Omega)$ and let φ be the indicator function of the closed, convex set $K = \{v \in W^{1,p}(\Omega) : v|_{\partial\Omega} \geq 0\}$, i.e.

$$\varphi(v) := \begin{cases} 0, & \text{if } v \in K \\ +\infty, & \text{if } v \notin K. \end{cases}$$

Then it is obvious that Φ is the indicator function of the closed convex set $\tilde{K} = \{v \in X : v|_{(0,T)\times\partial\Omega} \geq 0\}$. It is easy to see that the indicator function of a closed, convex set is convex lower semicontinuous. On the other hand, the identical zero function is an element of K, so $\varphi(0) = 0$ thus $(0,0) \in \partial\varphi$ which means that condition V2 is fulfilled. In addition, by choosing $V_0 = W_0^{1,p}(\Omega)$ condition V3 is also satisfied, since for arbitrary chosen $v \in W^{1,p}(\Omega)$ and $v_0 \in W_0^{1,p}(\Omega)$ we have $(v+v_0)|_{\partial\Omega} = v|_{\partial\Omega}$, thus $\varphi(v+v_0) = \varphi(v)$.

In the following, we show that by the above defined φ, N and f problem (2)–(3) might be considered as the weak form of a special parabolic mixed problem. If u is a sufficiently smooth solution of (2)–(3) then u is also a solution of the following problem

$$\begin{gathered} D_t u - N(u) - f = 0 \quad \text{in } Q_T, \\ u \geq 0, \quad \partial_{\nu^*}u \geq 0, \quad u\partial_{\nu^*}u = 0 \quad \text{in } (0,T)\times\partial\Omega \text{ and } u = u_0 \text{ in } \{0\}\times\Omega, \end{gathered} \tag{30}$$

In fact, let us suppose that u is a sufficiently smooth solution of the problem (2)–(3) with the above defined φ and A. Since $\tilde{K}$ is a cone with vertex origin therefore as in the first part of this section we obtain that

$$[Lu + A(u) - F, u] = 0, \qquad [Lu + A(u) - F, v] \geq 0 \quad \text{for all } v \in \tilde{K}. \tag{31}$$

This means that

$$\int_{Q_T} D_t u\, v\, dt\, dx + \int_{Q_T} \sum_{i=1}^n a_i(t,x,u,Du;u) D_i v\, dt\, dx + \int_{Q_T} a_0(t,x,u,Du;u) v\, dt\, dx \geq \int_{Q_T} f v\, dt\, dx$$

for all $v \in \tilde{K}$. In the second term by using Green's theorem we obtain

$$\begin{aligned} &\int_{Q_T} D_t u\, v\, dt\, dx - \int_{Q_T} \sum_{i=1}^n D_i\big[a_i(t,x,u,Du;u)\big] v\, dt\, dx + \\ &\qquad + \int_{Q_T} a_0(t,x,u,Du;u) v\, dt\, dx + \int_0^T \int_{\partial\Omega} \partial_{\nu^*}u\, v\, d\sigma_x dt \geq \int_{Q_T} f v\, dt\, dx. \end{aligned} \tag{32}$$

Choosing $v = \varphi \in C_0^\infty(Q_T) \in \tilde{K}$ in the above inequality we see that the differential equation $D_t u - N(u) - f = 0$ holds. By substituting this in (32) we obtain that $\int_0^T \int_{\partial\Omega} (\partial_{\nu^*}u) v \geq 0$ for all $v \in \tilde{K}$. As a consequence we have $\partial_{\nu^*}u \geq 0$ (because $v \in \tilde{K}$ means $v|_{(0,T)\times\partial\Omega} \geq 0$). The first part of (31) says that in inequality (32) with $v = u$ equality holds, thus $\int_0^T \int_{\partial\Omega} (\partial_{\nu^*}u) u = 0$. Since $u \in \tilde{K}$, $u|_{(0,T)\times\partial\Omega} \geq 0$, therefore $(\partial_{\nu^*}u) u|_{(0,T)\times\partial\Omega} = 0$. We obtained that u is a solution of problem (30), too. From the above argument it is obvious that if u is a solution of (30) then u is also a solution of (2)–(3).

In the above set K one can choose the general $\psi_1 \leq v|_{\partial\Omega} \leq \psi_2$ condition instead of $v|_{\partial\Omega} \geq 0$, where $\psi_1, \psi_2 \colon \partial\Omega \to \mathbb{R}$ are fixed functions such that $\psi_1 \leq 0$ and $\psi_2 \geq 0$ (thus $0 \in K$). By using the above arguments it is easy to see that in this case problem (2)–(3) might be considered as the weak form of the following parabolic problem:

$$\begin{gathered} D_t u - N(u) - f = 0 \text{ in } Q_T, \\ \psi_2 \geq u \geq \psi_1, \quad (u-\psi_1)\partial_{\nu^*}u = 0, \quad (u-\psi_2)\partial_{\nu^*}u = 0 \text{ in } (0,T)\times\partial\Omega, \text{ further,} \\ \text{if } u(t,x) = \psi_1(x) \text{ then } \partial_{\nu^*}u(t,x) \geq 0, \text{ if } u(t,x) = \psi_2(x) \text{ then } \partial_{\nu^*}u(t,x) \leq 0 \text{ for a.a. } (t,x) \text{ in } Q_T \\ u = u_0 \text{ in } \{0\}\times\Omega. \end{gathered}$$

Such problems may occur e.g. in elasticity theory.

4.2 Operator A

Suppose that function a_i $(i = 0, \dots, n)$ has the form:

$$a_i(t, x, \zeta_0, \zeta; v) = [H(v)](t, x) b_i(t, x, \zeta_0, \zeta) + [G(v)](t, x) d_i(t, x, \zeta_0, \zeta), \quad \text{if } i \neq 0, \text{ and} \tag{33}$$

$$a_0(t, x, \zeta_0, \zeta; v) = [H(v)](t, x) b_0(t, x, \zeta_0, \zeta) + [G_0(v)](t, x) d_0(t, x, \zeta_0, \zeta). \tag{34}$$

Then operators and functions b_i, d_i, H, G, G_0 may have e.g. the following form:

$$[H(v)](t, x) = \phi\left(\int_{Q_t} b(\tau, \xi) v(\tau, \xi)\, d\tau\, d\xi\right), \qquad [H(v)](t, x) = \phi\left(\left[\int_{Q_t} |v(\tau, \xi)|^\alpha\, d\tau\, d\xi\right]^{\frac{1}{\alpha}}\right),$$

where $b \in L^q(Q_T)$, $0 < \alpha \leq p$ and $\phi \colon \mathbb{R} \to \mathbb{R}$ is a continuous function such that there exists a constant $c > 0$ such that $\phi(\tau) \geq c$ for all $\tau \in \mathbb{R}$;

$$[G(v)](t, x) = \psi\left(\left|\int_0^t a(\tau, x) |v(\tau, x)|^\alpha\, d\tau\right|^{\frac{1}{\alpha}}\right), \qquad [G(v)](t, x) = \psi\left(\left|\int_\Omega a(t, \xi) |v(t, \xi)|^\alpha\, d\xi\right|^{\frac{1}{\alpha}}\right),$$

where $a \in L^\infty(Q_T)$ and $0 < \alpha \leq p$, further, $\psi \colon [0, \infty) \to \mathbb{R}$ is a nonegative continuous function such that for all $\tau \in [0, \infty)$ the estimation $|\psi(\tau)| \leq C \cdot |\tau|^{p-1-r_0}$ holds for some constants $C > 0$ and $0 < r_0 < p-1$; G_0 may have the same form as G, but we do not need the condition $\psi \geq 0$; finally

$$b_i(t, x, \zeta_0, \zeta) = \zeta_i |\zeta_i|^{p-2}\ (i = 0, \dots, n), \quad \text{or} \quad b_i(t, x, \zeta_0, \zeta) = \zeta_i |\zeta|^{p-2}\ (i \neq 0),\ b_0(t, x, \zeta_0, \zeta) = \zeta_0 |\zeta_0|^{p-2},$$

$$d_i(t, x, \zeta_0, \zeta) = \zeta_i |\zeta_i|^{r-1}\ (i = 0, \dots, n), \quad \text{or} \quad d_i(t, x, \zeta_0, \zeta) = \zeta_i |\zeta|^{r-1}\ (i \neq 0),\ d_0(t, x, \zeta_0, \zeta) = \zeta_0 |\zeta_0|^{r-1},$$

where $0 \leq r < r_0 < p - 1$. By using Hölder's inequality and elementary technics it is easy to verify that the above examples satisfy conditions F1–F5 (the detailed proof can be found in [2]).

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