ON UNIFORMLY MONOTONE OPERATORS ARISING IN NONLINEAR ELLIPTIC AND PARABOLIC PROBLEMS

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Dedicated to Professor László Simon on the occasion of his 70th birthday

ABSTRACT. We show some examples for uniformly monotone operators arising in weak formulation of nonlinear elliptic and parabolic problems. Besides the classical p-Laplacian some less known examples are given which are of interest because of applications.

1. INTRODUCTION

The aim of the present paper is to show several examples for uniformly monotone oprators arising in weak formulation of nonlinear elliptic and parabolic problems. Let X be a normed space and denote by X^* its dual, further, by $\langle \cdot, \cdot \rangle$ the pairing between X^* and X. Then, an operator $A\colon X \to X^*$ is called uniformly monotone (following the terminology of [7]) if there exist $p \geq 2$, $\gamma > 0$ such that

$$\langle A(u_1) - A(u_2), u_1 - u_2 \rangle \geq \gamma \cdot \|u_1 - u_2\|_X^p \tag{1.1}$$

for all $u_1, u_2 \in X$. In what follows, we study operators which are obtained by considering the weak formulation of an elliptic or parabolic equation or system with some boundary conditions, see, e.g., [6]. Namely, in the elliptic case let X be a linear subspace of $W^{1,p}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is bounded (with sufficiently smooth boundary), $p \geq 2$, and consider operator $A\colon X \to X^*$ defined by

$$\langle A(u), v \rangle = \int_\Omega \Big(\sum_{i=1}^n a_i(x, u(x), Du(x)) D_i v(x) + a_0(x, u(x), Du(x)) v(x) \Big)\, dx, \tag{1.2}$$

where D_i denotes the distributional derivative with respect to the i-th variable and $D = (D_1, \dots, D_n)$ is the gradient. The space X depends on the boundary conditions, for instance, $X = W^{1,p}(\Omega)$ in case of homogeneous Neumann type and $X = W_0^{1,p}(\Omega)$ (i.e. the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$) in case of homogeneous Dirichlet type condition (one can also have mixed boundary conditions, see [2]).

A weak form of an elliptic problem may be written as $A(u) = F$ where $F \in X^*$. In the simplest case

$$\langle F, v \rangle = \int_\Omega f(x) v(x)\, dx$$

with some $f \in L^q(\Omega)$ where $\frac{1}{p} + \frac{1}{q} = 1$.

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The parabolic case is a minor modification of the elliptic, X may be chosen as $L^p(0,T;V)$ (i.e., the set of measurable functions $u\colon (0,T)\to V$, see, e.g., [7]), where V is a linear subspace of $W^{1,p}(\Omega)$ and $0<T<\infty$, further, functions a_i may depend on variable t, and in (1.2) one integrates on $(0,T)\times\Omega$. The weak formulation of a parabolic problem may be written in the form $D_t u + Au = F$, where D_t denotes the distributional derivative with respect to the variable t.

Supposing the uniform monotonicity (and some other properties) of an operator of the form (1.2), one can prove uniqueness of solutions to the above abstract equations, continuous dependence of the solutions on data and for parabolic equations one can obtain results on asymptotic behavior as $t\to\infty$, see, e.g., [2, 6].

The well-known example for an operator having the form (1.2) is the following:

$$\begin{aligned} &a_i(x,\xi) = \xi_i|(\xi_1,\dots,\xi_n)|^{p-2} \quad (i=1,\dots,n),\\ &a_0(x,\xi) = \xi_0|\xi_0|^{p-2} \end{aligned}$$

where $\xi = (\xi_0,\xi_1,\dots,\xi_n)$ refers to $(u, D_1u,\dots,D_nu)$. In this case, assuming homogeneous Dirichlet boundary condition, Gauss's theorem yields that operator (1.2) may be considered as the weak form of the classical operator $u\mapsto -\Delta_p u + u|u|^{p-2}$ where

$$\Delta_p u = \operatorname{div}(\operatorname{grad} u|\operatorname{grad} u|^{p-2})$$

is the p-Laplacian. Note that in case $X = W_0^{1,p}(\Omega)$ operator $-\Delta_p$ is also uniformly monotone since, due to Poincaré's inequality, an equivalent norm can be introduced in $W_0^{1,p}(\Omega)$, see [1].

In [2, 3] we considered a nonlinear system consisting of three different types of differential equations: a first order ODE, a parabolic and an elliptic PDE. Such a system may occur, e.g., as a generalization of a model describing fluid flow in porous media. In that case operator (1.2) has a special form: functions a_i do not depend on $(\xi_0,\dots,\xi_k)$ if $i>k$. The present paper was motivated by such operators. Because of the application it is of interest to have some efficient criterion for uniform monotonicity and show some concrete uniform monotone operators of that type. In what follows, we shall give a variety of examples for uniformly monotone operators, including functions a_i of the above mentioned special type. In the next section we shall formulate and prove a result of [4] which is a sufficient condition on functions a_i for the uniform monotonicity of operator (1.2) and this will be applied to examples in Section 3. For further details on operators of monotone type, see [5, 7], for applications to parabolic and elliptic partial differential equations, see, [6].

2. A sufficient condition

Let $\Omega\subset\mathbb{R}^n$ be a bounded open subset and X be a linear subspace of $W^{1,p}(\Omega)$ ($p\ge 2$) and let us use the notations introduced in the previous section. We define operator $A\colon X\to X^*$ by the formula (1.2). Consider the abstract equation $A(u)=F$ where $F\in X^*$ (which may be obtained as a weak formulation of an elliptic boundary-value problem). Problems of this type have an extended classical theory (see, e.g., [5, 7]). Existence and uniqueness of solutions can be guaranteed by supposing the following well-known conditions:

(A1) The functions $a_i\colon \Omega\times\mathbb{R}^{n+1}\to\mathbb{R}$ ($i=0,\dots,n$) are of Carathéodory type, i.e., $a_i(x,\xi)$ is measurable in $x\in\Omega$ for all $\xi\in\mathbb{R}^{n+1}$, and continuous in $\xi\in\mathbb{R}^{n+1}$ for a.a. $x\in\Omega$.

(A2) There exist a constant $c > 0$ and a function $k \in L^q(\Omega)$ such that for a.a. $x \in \Omega$ and all $\xi \in \mathbb{R}^{n+1}$,

$$|a_i(x,\xi)| \le c \cdot |\xi|^{p-1} + k(x) \qquad (i = 0, \dots, n).$$

(A3) There exists a constant $C > 0$ such that for a.a. $x \in \Omega$ and all $\xi, \tilde{\xi} \in \mathbb{R}^n$,

$$\sum_{i=0}^{n} (a_i(x,\xi) - a_i(x,\tilde{\xi}))(\xi_i - \tilde{\xi}_i) \ge C \cdot |\xi - \tilde{\xi}|^p.$$

Clearly, after integration on Ω condition (A3) yields (1.1) for all $u_1, u_2 \in X$ with $\gamma = C$ thus (A3) ensures the uniform monotonicity of operator A. This implies uniqueness and continuous dependence on data of the solutions to $A(u) = F$. We note that existence of solutions may be shown also if condition (A3) is weakened as follows: the monotonicity is restricted to the main part of the operator and the so-called coercivity (in other words ellipticity) condition is assumed. In this case one obtains the pseudomonotonicity of operator A and then existence of solutions to $A(u) = F$ also follows, see [7].

Now we recall a result of [4] which is a sufficient condition on functions a_i guaranteeing condition (A3).

Proposition 2.1. *Suppose that $p \ge 2$ and a_i is continuously differentiable in variable ξ for all $i = 0, \dots, n$. Further, assume that there exists a constant $\delta > 0$ such that for a.a. $x \in \Omega$, all $\xi \in \mathbb{R}^{n+1}$ and all $(z_0, \dots, z_n) \in \mathbb{R}^{n+1}$,*

$$\sum_{j=0}^{n} \sum_{i=0}^{n} D_j a_i(x,\xi) z_i z_j \ge \delta \cdot \sum_{i=0}^{n} |\xi_i|^{p-2} z_i^2. \tag{2.1}$$

Then condition (A3) holds.

To prove this assertion we shall apply the following elementary inequality (2.2) from [4]. For convenience we present a proof of it below.

Lemma 2.2. *Let a, b be arbitrary and $s \ge 0$ real numbers. Then*

$$\int_0^1 |a + \tau b|^s \, d\tau \ge \frac{|b|^s}{2^s(s+1)}. \tag{2.2}$$

Proof. The case $b = 0$ is obvious otherwise by homogenity we may assume $b = 1$ and $a \le -\frac{1}{2}$. Then by elementary transformations

$$\int_0^1 |a+\tau|^s \, d\tau = \int_0^1 \left|\tilde{\tau} - \frac{1}{2}\right|^s d\tilde{\tau} + \int_1^{\frac{1}{2}-a} \left|\tilde{\tau} - \frac{1}{2}\right|^s d\tilde{\tau} - \int_0^{-\frac{1}{2}-a} \left|\tilde{\tau} - \frac{1}{2}\right|^s d\tilde{\tau}.$$

Clearly,

$$\int_1^{\frac{1}{2}-a} \left|\tilde{\tau} - \frac{1}{2}\right|^s d\tilde{\tau} \ge \int_0^{-\frac{1}{2}-a} \left|\tilde{\tau} - \frac{1}{2}\right|^s d\tilde{\tau},$$

thus

$$\int_0^1 |a+\tau|^s \, d\tau \ge \int_0^1 \left|\tau - \frac{1}{2}\right|^s d\tau = \frac{1}{2^s(s+1)}.$$

We see that the inequality (2.2) is sharp, equality holds if and only if $a = -\dfrac{b}{2}$. □

Now we prove Proposition 2.1. We follow the proof of [4].

Proof of Proposition 2.1. For fixed $x \in \Omega$, $\xi, \tilde{\xi} \in \mathbb{R}^{n+1}$ we define the functions $f_i \colon [0,1] \to \mathbb{R}$ by

$$f_i(\tau) = a_i(x, \tilde{\xi} + \tau(\xi - \tilde{\xi})), \quad i = 0, \ldots, n.$$

Then by applying assumption (2.1) and inequality (2.2) we may deduce

$$\begin{aligned}\sum_{i=0}^{n}(a_i(x,\xi) - a_i(x,\tilde{\xi}))(\xi_i - \tilde{\xi}_i) &= \sum_{i=0}^{n}(f_i(1) - f_i(0))(\xi_i - \tilde{\xi}_i) \\ &= \sum_{i=0}^{n}\int_0^1 \sum_{j=0}^{n} D_j a_i(\tilde{\xi} + \tau(\xi - \tilde{\xi}))(\xi_j - \tilde{\xi}_j)(\xi_i - \tilde{\xi}_i)\, d\tau \\ &\geq \delta \cdot \sum_{i=0}^{n}\int_0^1 |\tilde{\xi} + \tau(\xi - \tilde{\xi})|^{p-2}(\xi_i - \tilde{\xi}_i)^2 d\tau \\ &\geq \frac{\delta}{2^{p-2}(p-1)}|\xi - \tilde{\xi}|^p.\end{aligned}$$

Whence after integration we conclude

$$\langle A(u_1) - A(u_2), u_1 - u_2\rangle \geq \frac{\delta}{2^{p-2}(p-1)}\|u_1 - u_2\|_X^p.$$

Thus condition (A3) holds with $C = \dfrac{\delta}{2^{p-2}(p-1)}$. □

3. Examples

Now we show some examples of uniformly monotone operators which fulfil also conditions (A1), (A2). For simplicity, we consider examples not depending on variable x. In the sequel we always suppose $p \geq 2$.

Example 1 Let $a_i(\xi) = \xi_i|\xi_i|^{p-2}$ $(i = 0, \ldots, n)$. Then

$$\langle A(u), v\rangle = \int_\Omega \left(\sum_{i=1}^{n} D_i u D_i v |D_i u|^{p-2} + uv|u|^{p-2}\right) dx.$$

Note that functions a_i obviously fulfil conditions (A1), (A2). Now simple calculations yield $D_i a_i(\xi) = (p-1)|\xi_i|^{p-2}$ and $D_j a_i(\xi) = 0$ $(j \neq i)$. Hence

$$\sum_{j=0}^{n}\sum_{i=0}^{n} D_j a_i(\xi) z_i z_j = (p-1)\sum_{i=0}^{n}|\xi_i|^{p-2} z_i^2,$$

thus by Proposition 2.1 condition (A3) holds as well.

Example 2 Now let

$$\begin{aligned}&a_i(\xi) = \xi_i|(\xi_1, \ldots, \xi_n)|^{p-2} \quad (i = 1, \ldots, n), \\ &a_0(\xi) = \xi_0|\xi_0|^{p-2}.\end{aligned}$$

In this case

$$\langle A(u), v\rangle = \int_\Omega \Big(\sum_{i=1}^{n} D_i u D_i v |Du|^{p-2} + uv|u|^{p-2}\Big),$$

i.e., A is the weak form of operator $u \mapsto -\Delta_p u + u|u|^{p-2}$ mentioned in the introduction. Obviously, functions a_i satisfy conditions (A1), (A2). It is easy to see that

$$\begin{cases} D_j a_i(\xi) = (p-2)\xi_j\xi_i|(\xi_1,\dots,\xi_n)|^{p-4}, & \text{for } i,j>0,\ i\neq j;\\ D_i a_i(\xi) = |(\xi_1,\dots,\xi_n)|^{p-2} + (p-2)\xi_i^2|(\xi_1,\dots,\xi_n)|^{p-4}, & \text{for } i>0;\\ D_j a_0(\xi) = D_0 a_i(\xi) = 0, & \text{for } j>0, i>0;\\ D_0 a_0(\xi) = (p-1)|\xi_0|^{p-2}. \end{cases}$$

Hence

$$\begin{aligned}\sum_{j=0}^{n}\sum_{i=0}^{n} D_j a_i z_i z_j &= \sum_{i=1}^{n} |(\xi_1,\dots,\xi_n)|^{p-2} z_i^2 + (p-1)|\xi_0|^{p-2} z_0^2 \\ &\quad + (p-2)|(\xi_1,\dots,\xi_n)|^{p-4}\cdot \sum_{j=1}^{n}\sum_{i=1}^{n}\xi_i\xi_j z_i z_j \\ &= \sum_{i=1}^{n} |(\xi_1,\dots,\xi_n)|^{p-2} z_i^2 + (p-1)|\xi_0|^{p-2} z_0^2 \\ &\quad + (p-2)|(\xi_1,\dots,\xi_n)|^{p-4}\cdot \left(\sum_{i=1}^{n}\xi_i z_i\right)^2 \\ &\geq \sum_{i=0}^{n} |\xi_i|^{p-2} z_i^2,\end{aligned}$$

thus from Proposition 2.1 it follows that operator A is uniformly monotone.

Example 3 Let $a_i(\xi) = \xi_i|\xi|^{p-2} + g_i(\xi)$ $(i = 0,\dots,n)$, where the functions $g_i\colon \mathbb{R}^{n+1}\to\mathbb{R}$ are continuous, further, there exist positive constants c, ε such that

$$|g_i(\xi)| \leq c\cdot|\xi|^{p-1} \quad\text{and}\quad |D_j g_i(\xi)| \leq \frac{1}{n+1+\varepsilon}\cdot|\xi|^{p-2} \tag{3.1}$$

for all $\xi = (\xi_0,\dots,\xi_n)\in\mathbb{R}^{n+1}$ $(i,j = 0,\dots,n)$. It is clear that conditions (A1), (A2) hold. By using Example 2 and the inequality $|\alpha\beta| \leq \frac{1}{2}(\alpha^2+\beta^2)$ one obtains

$$\begin{aligned}\sum_{j=0}^{n}\sum_{i=0}^{n} D_j a_i(\xi) z_i z_j &\geq \sum_{i=0}^{n}|\xi|^{p-2}z_i^2 - \frac{1}{2}\sum_{j=0}^{n}\sum_{i=0}^{n}|D_j g_i(\xi)|(z_i^2+z_j^2) \geq \\ &\geq \sum_{i=0}^{n}|\xi|^{p-2}z_i^2 - (n+1)\sum_{i=0}^{n}\frac{1}{n+1+\varepsilon}|\xi|^{p-2}z_i^2 \\ &= \sum_{i=0}^{n}\frac{\varepsilon}{n+1+\varepsilon}|\xi|^{p-2}z_i^2,\end{aligned}$$

which implies condition (A3). As an example for functions g_i with the properties (3.1), consider, e.g.,

$$g_i(\xi) = \frac{1}{(n+1+\varepsilon)\cdot\max\{\alpha_j, j=0,\dots,n\}}\prod_{j=0}^{n}|\xi_j|^{\alpha_j}$$

where $\alpha_j \geq 0$ for all $j = 0,\dots,n$ and $\sum_{j=0}^{n}\alpha_j = p-1$.

Example 4 Now we show example for the system considered in [2] (which was mentioned in the Introduction). Suppose $2 \le p \le 4$, $1 \le k \le n$ and let

$$a_i(\xi) = \xi_i |\xi|^{p-2} \quad (0 \le i \le k \le n),$$
$$a_i(\xi) = \xi_i |(\xi_{k+1}, \dots, \xi_n)|^{p-2} \quad (k < i \le n).$$

We show that these functions fulfil condition (A3) ((A1) and (A2) obviously hold). Now for brevity let $\zeta = (\xi_{k+1}, \dots, \xi_n)$. Clearly,

$$\begin{cases} D_j a_i(\xi) = (p-2)\xi_i\xi_j|\xi|^{p-4}, & \text{for } 0 \le i \le k, 0 \le j \le n, j \ne i; \\ D_j a_i(\xi) = (p-2)\xi_i\xi_j|\zeta|^{p-4}, & \text{for } k < i \le n, k < j < n, j \ne i; \\ D_j a_i(\xi) = 0, & \text{for } k < i \le n, 0 \le j \le k \\ D_i a_i(\xi) = |\xi|^{p-2} + (p-2)\xi_i^2|\xi|^{p-4}, & \text{for } 0 \le i \le k; \\ D_i a_i(\xi) = |\zeta|^{p-2} + (p-2)\xi_i^2|\zeta|^{p-4}, & \text{for } k < i \le n. \end{cases}$$

Then

$$\begin{aligned} &\sum_{j=0}^{n}\sum_{i=0}^{n} D_j a_i(\xi) z_i z_j \\ &= \sum_{i=0}^{k} |\xi|^{p-2} z_i^2 + (p-2)|\xi|^{p-4} \sum_{j=0}^{n}\sum_{i=0}^{k} \xi_i\xi_j z_i z_j \\ &\quad + \sum_{i=k+1}^{n} |\zeta|^{p-2} z_i^2 + (p-2)|\zeta|^{p-4} \sum_{j=k+1}^{n}\sum_{i=k+1}^{n} \xi_i\xi_j z_i z_j \\ &= \sum_{i=0}^{k} |\xi|^{p-2} z_i^2 + \sum_{i=k+1}^{n} |\zeta|^{p-2} z_i^2 + (p-2)|\xi|^{p-4} \left(\sum_{i=0}^{k} \xi_i z_i\right)^2 \\ &\quad + (p-2)|\zeta|^{p-4} \left(\sum_{i=k+1}^{n} \xi_i z_i\right)^2 + (p-2)|\xi|^{p-4} \sum_{j=k+1}^{n}\sum_{i=0}^{k} \xi_i\xi_j z_i z_j. \end{aligned}$$

By using the estimate

$$\begin{aligned} \sum_{j=k+1}^{n}\sum_{i=0}^{k} \xi_i\xi_j z_i z_j &= \left(\sum_{j=k+1}^{n} \xi_j z_j\right)\left(\sum_{i=0}^{k} \xi_i z_i\right) \\ &\ge -\frac{1}{2}\left(\sum_{i=k+1}^{n} \xi_i z_i\right)^2 - \frac{1}{2}\left(\sum_{i=0}^{k} \xi_i z_i\right)^2. \end{aligned}$$

and the fact that $|\zeta|^{p-4} \ge |\xi|^{p-4}$ (since $p \le 4$) we conclude

$$\begin{aligned} &\sum_{j=0}^{n}\sum_{i=0}^{n} D_j a_i(\xi) z_i z_j \\ &= \sum_{i=0}^{k} |\xi|^{p-2} z_i^2 + \sum_{i=k+1}^{n} |\zeta|^{p-2} z_i^2 + \frac{1}{2}(p-2)|\xi|^{p-4} \left(\sum_{i=k+1}^{n} \xi_i z_i\right)^2 \\ &\quad + \frac{1}{2}(p-2)|\xi|^{p-4} \left(\sum_{i=0}^{k} \xi_i z_i\right)^2 \ge \sum_{i=0}^{n} |\xi_i|^{p-2} z_i^2. \end{aligned}$$

Now Proposition 2.1 yields condition (A3).

In case $p > 4$ one may consider, e.g., the following functions:

$$a_i(\xi) = \xi_i|(\xi_0,\dots,\xi_k)|^{p-2} + \xi_i|\xi|^{r-2} \quad (0 \le i \le k \le n),$$
$$a_i(\xi) = \xi_i|(\xi_{k+1},\dots,\xi_n)|^{p-2} + \xi_i|(\xi_{k+1},\dots,\xi_n)|^{r-2} \quad (k < i < n),$$

where $2 \le r \le 4$, $1 \le k \le n$. By using the previous examples it is not difficult to show that these functions satisfy condition (A3).

Example 5 Now let

$$a_i(\xi) = \xi_i|\xi_i|^{p-2} + \prod_{k=0}^{n} \xi_k|\xi_k|^{p-2} \cdot h_i(\xi) \quad (i = 0,\dots,n)$$

where functions $h_i\colon \mathbb{R}^{n+1} \to \mathbb{R}$ $(i = 0,\dots,n)$ are differentiable and have compact support S_i. Denote $S = \bigcup_{i=0}^{n} S_i$ and let

$$\alpha = p \max\left\{\sup_{\xi\in S} |\xi|^{(n+1)(p-1)}, 1\right\} \cdot \max\left\{\sup_S |h|, \sup_S |Dh|\right\}.$$

We show that if α is sufficiently small then functions a_i fulfil condition (A3) ((A1) is obvious and due to the compact supports (A2) is also satisfied). Observe that

$$D_i a_i(\xi) = (p-1)|\xi_i|^{p-2} + (p-1)|\xi_i|^{p-2} \prod_{k\ne i} \xi_k|\xi_k|^{p-2} \cdot h_i(\xi) + \prod_{k=0}^{n} \xi_k|\xi_k|^{p-2} \cdot D_i h(\xi),$$

thus

$$D_i a_i(\xi) \ge (p-1)|\xi_i|^{p-2} - \alpha|\xi_i|^{p-2}.$$

In addition, for $j \ne i$,

$$D_j a_i(\xi) = (p-1)|\xi_j|^{p-2} \prod_{k\ne j} \xi_k|\xi_k|^{p-2} \cdot h_i + \prod_{k=0}^{n} \xi_k|\xi_k|^{p-2} \cdot D_j h_i(\xi)$$

so that $|D_j a_i(\xi)| \le \alpha \cdot |\xi_i|^{p-2}$ and $|D_j a_i(\xi)| \le \alpha \cdot |\xi_j|^{p-2}$ hence

$$|D_j a_i(\xi) z_i z_j| \le \alpha \cdot \left(|\xi_i|^{p-2} z_i^2 + |\xi_j|^{p-2} z_j^2\right).$$

Therefore,

$$\begin{aligned}
\sum_{j=0}^{n}\sum_{i=0}^{n} D_j a_i(\xi) z_i z_j &\ge (p-1-\alpha) \sum_{i=0}^{n} |\xi_i|^{p-2} z_i^2 \\
&\quad - \alpha \sum_{j=0}^{n}\sum_{i=0}^{n} \left(|\xi_i|^{p-2} z_i^2 + |\xi_j|^{p-2} z_j^2\right) \\
&\ge (p-1-\alpha) \sum_{i=0}^{n} |\xi_i|^{p-2} z_i^2 - 2n\alpha \sum_{i=0}^{n} |\xi_i|^{p-2} z_i^2 \\
&= (p-1-(2n+1)\alpha) \sum_{i=0}^{n} |\xi_i|^{p-2} z_i^2.
\end{aligned}$$

Hence (2.1) holds provided α is sufficiently small which implies condition (A3).

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