A NOTE ON A TRACE INEQUALITY FOR POSITIVE BLOCK MATRICES

ÁDÁM BESENYEI

Abstract. We give a short proof of a trace inequality for 2×2 positive block matrices which is a special case of the subadditivity inequality for q-entropies.

1. Introduction

In this short note we prove the following remarkable inequality for positive-semidefinite block matrices.

Theorem. *Let $A, B, C \in \mathbb{C}^{n\times n}$ be such that the block matrix*

$$X = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathbb{C}^{2n\times 2n}$$

is positive-semidefinite. Then

$$\operatorname{Tr} AC - \operatorname{Tr} B^*B \leq \operatorname{Tr} A \operatorname{Tr} C - \operatorname{Tr} B^* \operatorname{Tr} B.$$

There holds equality if and only if $X = Y \otimes Z$ for some positive-semidefinite $Y \in \mathbb{C}^{2\times 2}$, $Z \in \mathbb{C}^{n\times n}$ such that $\min(\operatorname{rank} Y, \operatorname{rank} Z) \leq 1$.

Remark. Clearly, if the equality conditions hold, then both sides are 0.

The inequality of the Theorem comes from the subadditivity of q-entropies proved in [1]. It states that for any bipartite state ρ in a finite-dimensional Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ it holds

$$S_q(\rho) \leq S_q(\operatorname{Tr}_1 \rho) + S_q(\operatorname{Tr}_2 \rho)$$

where

$$S_q(\rho) = \frac{1 - \operatorname{Tr} \rho^q}{q-1} \quad (q > 1)$$

is the so-called q-entropy and the partial traces Tr_1 and Tr_2 are linear operators defined by $\operatorname{Tr}_1 \colon X \otimes Y \mapsto \operatorname{Tr}(X) Y$ and $\operatorname{Tr}_2 \colon X \otimes Y \mapsto \operatorname{Tr}(Y) X$ (see also [2]). If $\mathcal{H}_1 = \mathbb{C}^2$ and $\mathcal{H}_2 = \mathbb{C}^n$ and M_n denotes the space of $n \times n$ complex matrices, then for a density matrix

$$\rho = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in M_2 \otimes M_n$$

the partial traces can be expressed as

$$\operatorname{Tr}_1 \rho = A + C \in M_n \quad \text{and} \quad \operatorname{Tr}_2 \rho = \begin{bmatrix} \operatorname{Tr} A & \operatorname{Tr} B \\ \operatorname{Tr} B^* & \operatorname{Tr} C \end{bmatrix} \in M_2.$$

Date: January 14, 2013.
2010 *Mathematics Subject Classification.* 15A45.
Key words and phrases. positive block matrix, trace, entropy, inequality.

So for $q = 2$ the subadditivity inequality has the form

$$-\operatorname{Tr}(A^2+C^2)-2\operatorname{Tr}B^*B \le 1-\operatorname{Tr}(A+C)^2-(\operatorname{Tr}A)^2-(\operatorname{Tr}C)^2-2\operatorname{Tr}B^*\operatorname{Tr}B$$

which reduces to the inequality of the Theorem by using $\operatorname{Tr}\rho = \operatorname{Tr}(A+C) = 1$. The proof of the subadditivity relation in [1] is a bit delicate and therefore the inequality of the Theorem deserves to have an elementary proof. This will be provided in the following.

2. Proof of the inequality

We may assume that A is diagonal. Indeed, if $U^*AU = \Lambda$ with unitary U and diagonal Λ, then

$$\begin{bmatrix} U^* & 0 \\ 0 & U^* \end{bmatrix} \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix} = \begin{bmatrix} U^*AU & U^*BU \\ U^*B^*U & U^*CU \end{bmatrix}$$

and

$$\operatorname{Tr}(U^*AU)(U^*CU) - \operatorname{Tr}(U^*B^*U)(U^*BU) = \operatorname{Tr}AC,$$
$$\operatorname{Tr}(U^*AU)\operatorname{Tr}(U^*CU) - \operatorname{Tr}(U^*B^*U)\operatorname{Tr}(U^*BU) = \operatorname{Tr}A\operatorname{Tr}C - \operatorname{Tr}B^*\operatorname{Tr}B.$$

For diagonal A, the inequality has the form

$$\sum_{i=1}^n a_{ii}c_{ii} - \sum_{i,j=1}^n |b_{ij}|^2 \le \sum_{i=1}^n a_{ii} \cdot \sum_{i=1}^n c_{ii} - \left| \sum_{i=1}^n b_{ii} \right|^2$$

which can be simplified to

$$2\sum_{i>j} \Re(\overline{b_{ii}}b_{jj}) - \sum_{i\neq j} |b_{ij}|^2 \le \sum_{i>j} (a_{ii}c_{jj} + a_{jj}c_{ii}).$$

Since the block matrix $X = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ is positive-semidefinite, A and C are also positive-semidefinite. Further, the principal subdeterminants of X, especially the 2×2 subdeterminants are non-negative (see [3]) thus $a_{ii}c_{ii} \ge |b_{ii}|^2$ for $i = 1,\dots,n$. Therefore, by the inequality of the arithmetic and geometric means we obtain

$$a_{ii}c_{jj} + a_{jj}c_{ii} \ge 2\sqrt{a_{ii}c_{ii}a_{jj}c_{jj}} \ge 2\sqrt{|b_{ii}|^2|b_{jj}|^2} \ge 2\Re(\overline{b_{ii}}b_{jj})$$

and the desired inequality follows.

3. The case of equality

If $n = 1$, there holds equality. Suppose that $n \ge 2$ and A is diagonal. From the above arguments it follows that in the case of equality the following conditions must hold for $i, j = 1,\dots,n, i \neq j$:

(i) $a_{ii}c_{ii} \ge |b_{ii}|^2$ and $a_{ii}c_{ii}a_{jj}c_{jj} = |b_{ii}|^2|b_{jj}|^2$;
(ii) $a_{ii}c_{jj} = a_{jj}c_{ii}$;
(iii) $b_{ij} = 0$;
(iv) $|b_{ii}b_{jj}| = \Re(\overline{b_{ii}}b_{jj})$.

Clearly, (iii) implies that B is diagonal. Further, (iv) means that the numbers b_{ii} have the same argument.

If $a_{ii}c_{ii} > |b_{ii}|^2$ for some i, then by (i), $a_{jj}c_{jj} = |b_{jj}|^2 = 0$ for all $j \neq i$. So by (ii), $a_{jj} = c_{jj} = 0$ and also $b_{jj} = 0$ for $j \neq i$. But C must be positive-semidefinite thus $c_{ii}c_{jj} \geq |c_{ij}|^2$ $(j \neq i)$ so C should also be diagonal with at most one non-zero diagonal entry, similarly to A and B. This means that $X = Y \otimes Z$ where $Y = \begin{bmatrix} a_{ii} & b_{ii} \\ \overline{b_{ii}} & c_{ii} \end{bmatrix}$ is positive-definite and Z is diagonal with one non-zero entry, $z_{ii} = 1$.

Now assume that $a_{ii}c_{ii} = |b_{ii}|^2$ for all i Since X is positive-semidefinite, the following 3×3 subdeterminant should be non-negative:

$$\begin{vmatrix} a_{ii} & 0 & b_{ii} \\ 0 & c_{jj} & c_{ji} \\ \overline{b_{ii}} & \overline{c_{ji}} & c_{ii} \end{vmatrix} = \left(a_{ii}c_{ii} - |b_{ii}|^2\right) c_{ij} - a_{ii}|c_{ij}|^2 = -a_{ij}|c_{ij}|^2.$$

We have two cases. If $A = 0$, then $B = 0$ and C is an arbitrary positive-semidefinite matrix thus $X = Y \otimes Z$ where $Y = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Otherwise $c_{ij} = 0$ $(j \neq i)$ when $a_{ii} \neq 0$. Moreover, by (ii), $c_{ii} = 0$ when $a_{ii} = 0$ and thus by the positive-semidefiniteness of C, $c_{ij} = 0$ $(j \neq i)$. So C is also diagonal, proportional to A and therefore by (iv), $B = \lambda A, C = |\lambda|^2 A$ for some constant $\lambda \in \mathbb{C}$. Thus $X = Y \otimes A$ where $Y = \begin{bmatrix} 1 & \lambda \\ \overline{\lambda} & |\lambda|^2 \end{bmatrix}$. Then X is positive-semidefinite since it is the tensor product of positive-semidefinite matrices.

In the general case, if A is not diagonal, then we might consider

$$\tilde{X} = \begin{bmatrix} U^*AU & U^*BU \\ U^*B^*U & U^*CU \end{bmatrix}.$$

In the case of equality $\tilde{X} = \tilde{Y} \otimes \tilde{Z}$ thus

$$X = (I \otimes U)(\tilde{Y} \otimes \tilde{Z})(I \otimes U^*) = \tilde{Y} \otimes (U\tilde{Z}U^*)$$

where $\operatorname{rank} U\tilde{Z}U^* = \operatorname{rank} \tilde{Z}$.

References

[1] K. Audenaert, Subadditivity of q-entropies for $q > 1$, *J. Math. Phys.*, **48**, 083507 (2007).

[2] R. Bhatia, Partial traces and entropy inequalities, *Linear Algebra Appl.*, **375** (2003), 211–220.

[3] R. A. Horn, Ch. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1990.

Ádám Besenyei

Department of Applied Analysis, Eötvös Loránd University, H-1117 Budapest, Pázmány P. sétány 1/C, Hungary

E-mail address: badam@cs.elte.hu