On nonlinear systems containing nonlocal terms

Summary of the PhD Dissertation

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1 Introduction

Our aim is to give an overview of the author's dissertation. In the following we shall present mostly the motivation and the main points of our investigations. For rigorous explanation of the results we always refer to the dissertation and the references therein.

We study systems of nonlinear parabolic differential equations containing nonlocal terms, in other words, functional differential equations. By "nonlocal term" we mean terms which may depend not only on the value of the unknown at a certain point but also on values at other points, for example, it may contain a delay or an integral of the unknown on a domain etc. Such problems may occur in some physical models. For instance, in some diffusion processes the diffusion coefficient may depend on a nonlocal quantity, e.g., in population dynamics the growing rate of a population may depend on the size of the population, mathematically, on the integral of the density, see [10, 11]. We mention two other important applications. First, climatology, see, e.g., [13]. Second, modelling of fluid flow, especially in porous media, see [12, 16]. For other nonlocal models such as transmission problems, or nonlocal boundary conditions, see the references of the dissertation. We note that instead of equations one may consider nonlocal variational inequalities. That type of problems occur in elasticity theory, see [4, 14].

In the following, we consider two systems of differential equations containing nonlocal terms. The first one consists of parabolic functional equations of general divergence form. The second one is a generalization of a system describing fluid flow in porous media and consists of three different types of differential equations.

The main tool of our further investigations will be the theory of operators of monotone type. For a detailed introduction to this theory and its applications, see [8, 15, 20]. In particular, we shall apply some results of [7, 9] related to pseudomonotone operators.

Existence of weak solutions in time interval $(0, T)$ $(0 < T \leq \infty)$ is shown for both systems, further, asymptotic properties are studied such as the boundedness and stabilization (i.e., convergence to equilibrium) of solutions. For examples illustrating our results we refer to our dissertation. It is worth mentioning the monographs [17, 19] which consider functional differential equations by means of semigroups.

2 Notation

Throughout this paper, $\Omega \subset \mathbb{R}^n$ will be a bounded domain with smooth boundary (e.g., C^1 is sufficient) and $0 < T < \infty$. We write briefly $Q_T = (0, T) \times \Omega$ and $Q_\infty = (0, \infty) \times \Omega$. In the sequel, $W^{1,p}(\Omega)$, $W_0^{1,p}(\Omega)$ denotes the usual Sobolev spaces, further, $L^p(0, T; V)$ will be the set of measurable functions $u \colon (0, T) \to V$ such that $\|u\|_{L^p(0,T;V)} := \left(\int_0^T \|u\|_V^p \right)^{\frac{1}{p}} < \infty$

$(1 < p < \infty)$. It is well-known that $(L^p(0,T;V))^* = L^q(0,T;V^*)$ where $\frac{1}{p} + \frac{1}{q} = 1$. In addition, $L^p_{\mathrm{loc}}(0,\infty;V)$ is the set of measurable functions $u\colon (0,\infty) \to V$ such that $u|_{(0,T)} \in L^p(0,T;V)$ for every $0 < T < \infty$. The pairing between $(W^{1,p}(\Omega))^*$ and $W^{1,p}(\Omega)$, further, between $L^q(0,T;V^*)$ and $L^p(0,T;V)$ is denoted by $\langle \cdot, \cdot \rangle$, $[\cdot, \cdot]$, respectively. Finally, D_i, D_t stand for the distributional differentiation with respect to x_i and t, respectively, and $D = (D_1, \ldots, D_n)$.

3   A system of parabolic equations

Consider the following parabolic differential equation containing nonlocal term:

$$D_t u(t,x) - \mathrm{div}\left(g\Big(\int_\Omega u(t,x)\,dx \Big) Du(t,x) \right) = f(t,x) \tag{1}$$

for $t > 0$, $x \in \mathbb{R}^n$ where functions $f\colon (0,\infty) \times \mathbb{R}^n \to \mathbb{R}$, $g\colon \mathbb{R} \to \mathbb{R}$ are given and $u\colon (0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is the unknown with initial condition $u(0,x) = \varphi(x)$ for $x \in \mathbb{R}^n$. Such equation is motivated, e.g., by diffusion processes for heat or population. In [10, 11] quasilinear equations similar to (1) were considered and existence of solutions, further, asymptotic properties of solutions were shown. In [18] quasilinear parabolic functional equations of general divergence form were investigated by means of monotone operators. Existence of weak solutions and some qualitative properties of weak solutions were proved. These results are extended to systems of nonlocal parabolic equations in the first part of the dissertation which will be now briefly summarized (see [1]).

Consider the following system containing N nonlocal parabolic differential equations:

$$\begin{aligned}
D_t u^{(l)}(\cdot) - \sum_{i=1}^{n} D_i &\left[a_i^{(l)}(\cdot, u^{(1)}(\cdot), \ldots, u^{(N)}(\cdot), Du^{(1)}(\cdot), \ldots, Du^{(N)}(\cdot); u^{(1)}, \ldots, u^{(N)}) \right] \\
&+ a_0^{(l)}(\cdot, u^{(1)}(\cdot), \ldots, u^{(N)}(\cdot), Du^{(1)}(\cdot), \ldots, Du^{(N)}(\cdot); u^{(1)}, \ldots, u^{(N)}) = f^{(l)}(\cdot)
\end{aligned} \tag{2}$$

where $(\cdot)$ refers to the variable $(t,x) \in Q_T$ and the terms after the symbol ";" represent the nonloacal variables $(l = 1, \ldots, N)$. We may pose, for simplicity, homogeneous initial condition, further, boundary conditions of homogeneous Dirichlet or Neumann type.

Fix $p \geq 2$ and denote by V a closed linear subspace of $(W^{1,p}(\Omega))^N$ (determined by the boundary condition, e.g., in case of homogeneous Neumann type $V = (W^{1,p}(\Omega))^N$, in case of homogeneous Dirichlet type $V = (W_0^{1,p}(\Omega))^N$). Let $X = L^p(0,T;V)$ which will be the space of weak solutions. A function $v \in X$ has its coordinate-functions $(v^{(1)}, \ldots, v^{(N)})$, vectors $\xi \in \mathbb{R}^{(n+1)N}$ will have the form $\xi = (\zeta_0, \zeta)$ where $\zeta_0 = (\zeta_0^{(1)}, \ldots, \zeta_0^{(N)}) \in \mathbb{R}^N$ and $\zeta = (\zeta_1^{(1)}, \ldots, \zeta_1^{(N)}, \ldots, \zeta_n^{(1)}, \ldots, \zeta_n^{(N)}) \in \mathbb{R}^{nN}$ (the sub-indeces indicate the variable of differentiation, the super-indeces the actual coordinate-function).

We pose some natural assumptions on the above functions $a_i^{(l)}$ to obtain existence of weak solutions in $(0,T)$ to system (2). For $i = 0, \ldots, n$; $l = 1, \ldots, N$, suppose:

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(A1) Function $a_i^{(l)}\colon Q_T \times \mathbb{R}^{(n+1)N} \times X \to \mathbb{R}$ has the Carathéodory property for every fixed $v \in X$, i.e., it is measurable in (t,x) for every $(\zeta_0, \zeta) \in \mathbb{R}^{(n+1)N}$ and continuous in (ζ_0, ζ) for a.a. $(t,x) \in Q_T$.

(A2) There exist bounded operators $g_1\colon X \to \mathbb{R}^+$ and $k_1\colon X \to L^q(Q_T)$ such that $|a_i^{(l)}(t,x,\zeta_0,\zeta;v)| \leq g_1(v)\,(|\zeta_0|^{p-1} + |\zeta|^{p-1}) + [k_1(v)](t,x)$ holds for a.a. $(t,x) \in Q_T$, every $(\zeta_0, \zeta) \in \mathbb{R}^{(n+1)N}$ and $v \in X$.

(A3) For a.a. $(t,x) \in Q_T$, every $\zeta \neq \tilde\zeta \in \mathbb{R}^{nN}$, $\zeta_0 \in \mathbb{R}^N$ and $v \in X$,

$$\sum_{l=1}^{N} \sum_{i=1}^{n} \left(a_i^{(l)}(t,x,\zeta_0,\zeta;v) - a_i^{(l)}(t,x,\zeta_0,\tilde\zeta;v) \right) (\zeta_i^{(l)} - \tilde\zeta_i^{(l)}) > 0.$$

(A4) There exist operators $g_2\colon X \to \mathbb{R}^+$ and $k_2\colon X \to L^1(Q_T)$ such that for a.a. $(t,x) \in Q_T$, every $(\zeta_0, \zeta) \in \mathbb{R}^{(n+1)N}$ and $v \in X$,

$$\sum_{l=1}^{N} \sum_{i=0}^{n} a_i^{(l)}(t,x,\zeta_0,\zeta;v)\zeta_i^{(l)} \geq g_2(v)\,(|\zeta_0|^p + |\zeta|^p) - [k_2(v)](t,x),$$

further, $\displaystyle\lim_{\|v\|_X \to \infty} \left(g_2(v)\|v\|_X^{p-1} - \|k_2(v)\|_{L^1(Q_T)}\|v\|_X^{-1} \right) = +\infty.$

(A5) If $u_k \to u$ weakly in X and strongly in $L^p(0,T;(L^p(\Omega))^N)$ then

$$\lim_{k \to \infty} \|a_i^{(l)}(\,\cdot\,, u_k(\cdot), Du_k(\cdot); u_k) - a_i^{(l)}(\,\cdot\,, u_k(\cdot), Du_k(\cdot); u)\|_{L^q(Q_T)} = 0.$$

Conditions (A1)–(A4) are similar to the classical case when there is no nonlocality, see [9, 15, 20], (A2)–(A4) represent growth, monotonicity, coercivity. Besides these, (A5) means a kind of "continuity" in the nonlocal variable.

Now let us introduce operator $A\colon X \to X^*$ as follows. For $u,v \in X$ define

$$[A(u), v] := \sum_{l=1}^{N} \int_{Q_T} \left(\sum_{i=1}^{n} a_i^{(l)}(u, Du; u)D_i v^{(l)} + a_0^{(l)}(u, Du; u)v^{(l)} \right).$$

Further, let $D(L) = \{u \in X\colon D_t u \in X^*,\ u(0) = 0\}$ and define $L\colon D(L) \to X^*$ as $Lu = D_t u$. Finally, let $F \in X^*$. By the operators above the weak form of system (2) in $(0,T)$ is

$$Lu + A(u) = F. \tag{3}$$

By applying the theory of pseudomonotone operators (see [7]) we may prove

Theorem 3.1. *Assume that conditions (A1)–(A5) hold. Then operator $A\colon X \to X^*$ is bounded, demicontinuous, coercive and pseudomonotone with respect to $D(L)$. Consequently, for every $F \in X^*$ there exists a solution $u \in X$ of problem (3).*

It is not so difficult to show existence of weak solutions to (2) in $(0, \infty)$. Let $X^\infty = L^p_{\text{loc}}(0, \infty; V)$ which will be the space of weak solutions in $(0, \infty)$ and assume

(Vol) The restrictions $a_i^{(l)}(t, x, \zeta_0, \zeta, \eta_0, \eta; v)|_{(0,T)}$ of functions $a_i^{(l)} \colon Q_\infty \times \mathbb{R} \times \mathbb{R}^{n+1} \times X^\infty \to \mathbb{R}$ $(i = 0, \ldots, n; \; l = 1, \ldots, N)$ depend only on $v|_{(0,T)}$ for every $0 < T < \infty$.

The above condition is called Volterra property which means, roughly speaking, that the nonlocality does not depend on the future. Now by using a "diagonal method" one obtains

Theorem 3.2. *Assume (Vol) and suppose that conditions (A1)–(A5) hold in $(0, \infty)$ in the sense that they are satisfied by the restrictions of functions $a_i^{(l)}$ to $(0, T)$ for all $0 < T < \infty$. Then there exists $u \in X^\infty$ which is a weak solution of (2) in $(0, \infty)$ in the sense that $u|_{(0,T)}$ is a solution of problem (3) for every $0 < T < \infty$.*

Boundedness of solutions in $(0, \infty)$ follows by posing extra conditions on coercivity:

(A4*) There exist a constant $g_2 \in \mathbb{R}^+$ and a Volterra operator $k_2 \colon X^\infty \to L^1_{\text{loc}}(Q_\infty)$ such that for a.a. $(t, x) \in Q_\infty$, every $(\zeta_0, \zeta) \in \mathbb{R}^{(n+1)N}$ and $v \in X^\infty$,

$$\sum_{l=1}^{N} \sum_{i=0}^{n} a_i^{(l)}(t, x, \zeta_0, \zeta; v) \zeta_i^{(l)} \geq g_2 \left(|\zeta_0|^p + |\zeta|^p \right) - [k_2(v)](t, x).$$

In addition, there exist constants $c_4 > 0$, $0 \leq p_1 < p$ and a function $\varphi \in C(\mathbb{R}^+)$ such that $\lim_{\tau \to \infty} \varphi(\tau) = 0$, further, if $v \in X^\infty$ and $D_t v \in L^q_{\text{loc}}(0, \infty; V^*)$ then for a.a. $t > 0$,

$$\int_\Omega |[k_2(v)](t, x)| \, dx \leq c_4 \left(\sup_{\tau \in [0,t]} \|v(\tau)\|_{(L^2(\Omega))^N}^{p_1} + \varphi(t) \cdot \sup_{\tau \in [0,t]} \|v(\tau)\|_{(L^2(\Omega))^N}^{p} + 1 \right).$$

Theorem 3.3. *Assume (Vol), further, conditions (A1)–(A5) are satisfied in $(0, \infty)$ (in the same sense as in Theorem 3.2) with extra assumption (A4*), further, $F \in L^q_{\text{loc}}(0, \infty; V^*)$. Then $u \in L^\infty(0, \infty; (L^2(\Omega))^N)$ for the solutions u formulated in Theorem 3.2.*

With some further assumptions stabilization of solutions as $t \to \infty$ follows.

(A2+) For every fixed $v \in X^\infty \cap L^\infty(0, \infty; (L^2(\Omega))^N)$ there exists $c_v > 0$ and $k_v \in L^q(\Omega)$ such that $|a_i^{(l)}(t, x, \zeta_0, \zeta; v)| \leq c_v \left(|\zeta_0|^{p-1} + |\zeta|^{p-1} \right) + k_v(x)$ for a.a. $x \in \Omega$ and every $(\zeta_0, \zeta) \in \mathbb{R}^{(n+1)N}$ $(i = 0, \ldots, n; \; l = 1, \ldots, N)$.

(A6) There exist Carathéodory functions $a_{i,\infty}^{(l)} \colon \Omega \times \mathbb{R}^{(n+1)N} \to \mathbb{R}$ such that for every fixed $v \in X^\infty \cap L^\infty(0, \infty; (L^2(\Omega))^N)$, for a.a. $x \in \Omega$ and every $(\zeta_0, \zeta) \in \mathbb{R}^{(n+1)N}$, $\lim_{t \to \infty} a_i^{(l)}(t, x, \zeta_0, \zeta; v) = a_{i,\infty}^{(l)}(x, \zeta_0, \zeta)$.

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(A7) There exists $c_5 > 0$ such that for a.a. $x \in \Omega$, every $(\zeta_0, \zeta), (\tilde{\zeta}_0, \tilde{\zeta}) \in \mathbb{R}^{(n+1)N}$, $v \in X^\infty$,

$$\sum_{l=1}^{N} \sum_{i=0}^{n} \left(a_i^{(l)}(t, x, \zeta_0, \zeta; v) - a_i^{(l)}(t, x, \tilde{\zeta}_0, \tilde{\zeta}; v) \right) \left(\zeta_i^{(l)} - \tilde{\zeta}_i^{(l)} \right)$$

$$\geq c_5 \left(|\zeta_0 - \tilde{\zeta}_0|^p + |\zeta - \tilde{\zeta}|^p \right) - k_3(t, x, \zeta_0, \tilde{\zeta}_0; v)$$

where $\displaystyle\lim_{t \to \infty} \int_\Omega k_3(t, x, u(t, x), \tilde{u}(t, x); v) \, dx = 0$ if $u, \tilde{u}, v \in L^\infty(0, \infty; (L^2(\Omega))^N)$.

Note that (A6) is a natural assumption that means the stabilization of the functions $a_i^{(l)}$ as $t \to \infty$, further, (A7) is a uniform monotonicity type condition which provides existence of a unique equilibrium state. Now define operator $A_\infty : V \to V^*$, for $v, w \in V$ let

$$\langle A_\infty(v), w \rangle := \sum_{l=1}^{N} \int_\Omega \sum_{i=1}^{n} \left(a_{i,\infty}^{(l)}(v, Dv) D_i w^{(l)} + a_{0,\infty}^{(l)}(v, Dv) w^{(l)} \right).$$

Theorem 3.4. *Assume (Vol) and suppose that (A1)–(A7) are satisfied in $(0, \infty)$ (in the same sense as in Theorem 3.2) and there exists $F_\infty \in V^*$ such that $\displaystyle\lim_{t \to \infty} \|F(t) - F_\infty\|_{V^*} = 0$. Then there exists a unique $u_\infty \in V$ such that $A_\infty(u_\infty) = F_\infty$. In addition, if u is a solution formulated in Theorem 3.2 then $\displaystyle\lim_{t \to \infty} \|u(t) - u_\infty\|_{(L^2(\Omega))^N} = 0$.*

By posing polynomial estimates on the "speed" of the convergences in condition (A6), one may obtain polynomial estimates for the "speed" of the convergences stated in Theorem 3.4. For some examples satisfying the conditions of the above theorems, see [1, 5].

For system (2) it is not quite clear how to define the notion of periodic solutions, however, for a modification of (2) it is possible and one may show existence of them.

4 A system containing three types of equations

The second part of the dissertation is devoted to the investigation of a system which consists of there different types of differential equations: an ordinary, a parabolic and an elliptic one. This kind of problem is motivated by a fluid flow model in porous medium. A porous medium is a solid medium with lots of tiny holes (e.g., limestone). The flow of a fluid through the medium is determined by the large surface of the solid matrix and the closeness of the holes. If the fluid carries chemical species, chemical reactions can occur. Among these include reactions that can change the porosity. This process was modelled in [16] by the following system in one dimension:

$$\omega(\cdot) D_t u(\cdot) = D_x \alpha(|v(\cdot)| u_x(\cdot)) + K(\omega(\cdot)) D_x p(\cdot) u_x(\cdot) - ku(\cdot) g(\omega(\cdot)) \tag{4}$$

$$D_t \omega(\cdot) = bu(\cdot) g(\omega(\cdot)) \tag{5}$$

$$D_x(K(\omega(\cdot)) D_x p(\cdot)) = bu(\cdot) g(\omega(\cdot)), \tag{6}$$

$$v(\cdot) = -K(\omega(\cdot)) D_x p(\cdot) \tag{7}$$

in $(0, \infty) \times (0,1)$ with initial and boundary conditions $u(0,x) = u_0(x)$, $\omega(0,x) = \omega_0(x)$ for $x \in (0,1)$, $u(t,0) = u_1(t)$, $D_x u(t,1) = 0$ for $t > 0$ and $p(t,0) = 1$, $p(t,1) = 0$ for $t > 0$. where ω is the porosity, u is the concentration of the dissolved chemical solute carried by the fluid, p is the pressure, v is the velocity, further, α, k, b are given constants, K and g are given real functions. Observe that v is explicitly given by ω and p in equation (7) thus we may eliminate equation (7) by substituting it into (4). Further, for fixed u equation (5) is an ordinary differential equation with respect to the function ω; for fixed ω and p equation (4) is of parabolic type with respect to the function u; and for fixed ω and u equation (6) is of elliptic type with respect to the function p. This argument shows that the above system is a hybrid evolutionary/elliptic problem. In [12] a similar model was considered by using the method of Rothe.

The following generalization of the above system was considered in the second part of the dissertation (see [2, 3, 6]):

$$D_t\omega(t,x) = f(t,x,\omega(t,x),u(t,x);u), \qquad \omega(0,x) = \omega_0(x), \tag{8}$$

$$D_t u(t,x) - \sum_{i=1}^{n} D_i\left[a_i(t,x,\omega(t,x),u(t,x),Du(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p})\right] \tag{9}$$
$$+ a_0(t,x,\omega(t,x),u(t,x),Du(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p}) = g(t,x),$$

$$\sum_{i=1}^{n} D_i[b_i(t,x,\omega(t,x),u(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p})] \tag{10}$$
$$+ b_0(t,x,\omega(t,x),u(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p}) = h(t,x)$$

with initial conditions $\omega(0,x) = \omega_0(x)$, $u(0,x) = 0$ and boundary conditions of homogeneous Dirichlet or Neumann type ($\mathbf{p}$ is written by boldface letter in order to distinguish it from exponents). Existence and some qualitative properties of weak solutions is proved by using the theory of operators of monotone type (see [2, 3, 6]). The main idea is twofold. First the choice of the spaces for the weak solutions. Second, the idea of the proof which is to apply the so-called successive approximation. We now briefly sketch the results.

Let $2 \leq p_1, p_2 < \infty$, V_i a closed linear subspace of $W^{1,p_i}(\Omega)$ (depending on the boundary conditions), $X_i = L^{p_i}(0,T;V_i)$ ($i = 1, 2$) which will be the space of weak solutions u and $\mathbf{p}$, respectively. According to its original physical meaning, the space of ω will be $L^\infty(Q_T)$.

Now we sketch the assumptions made on functions a_i, b_i analogously to Section 3.

(A) Functions a_i: Carathéodory, growth, "monotonicity", coercivity, "continuity" in the nonlocal variables; conditions determined by exponent p_1.

(B) Functions b_i: Carathéodory, growth, "uniform monotonicity", coercivity, "continuity" in the nonlocal variables; conditions determined by exponent p_2.

(F) Function f: Carathéodory, Lipschitz, "continuity" in the nonlocal variable, "sign" condition (attractive steady-state).

Now define operators $A\colon L^\infty(Q_T) \times X_1 \times X_2 \to X_1^*$, $B\colon L^\infty(Q_T) \times X_1 \times X_2 \to X_2^*$ by

$$[A(\omega, u, \mathbf{p}), v_1] = \int_{Q_T} \left(\sum_{i=1}^n a_i(\omega, u, Du, \mathbf{p}, D\mathbf{p}; \omega, u, \mathbf{p}) D_i v_1 + a_0(\omega, u, Du, \mathbf{p}, D\mathbf{p}; \omega, u, \mathbf{p}) v_1 \right),$$

$$[B(\omega, u, \mathbf{p}), v_2] = \int_{Q_T} \left(\sum_{i=1}^n b_i(\omega, u, \mathbf{p}, D\mathbf{p}; \omega, u, \mathbf{p}) D_i v_2 + b_0(\omega, u, \mathbf{p}, D\mathbf{p}; \omega, u, \mathbf{p}) v_2 \right),$$

for $v_i \in X_i$ $(i = 1, 2)$. In addition, let $L : D(L) \to X_1^*$ be the operator of differentiation $Lu = D_t u$ with its domain $D(L) = \{u \in X_1 \colon D_t u \in X_1^*, \ u(0) = 0\}$.

By the operators above, the weak form of system (8)–(10) in $(0, T)$ is defined as

$$\omega(t, x) = \omega_0(x) + \int_0^t f(s, x, \omega(s, x), u(s, x); u)\, ds \quad \text{a.e. in } Q_T \tag{11}$$

$$Lu + A(\omega, u, \mathbf{p}) = G \tag{12}$$

$$B(\omega, u, \mathbf{p}) = H. \tag{13}$$

where $G \in X_1^*$, $H \in X_2^*$. Our result on existence of solutions is (see [2, 6])

Theorem 4.1. *Suppose that conditions (A), (B), (F) hold. Then for every $\omega_0 \in L^\infty(\Omega)$, $G \in X_1^*$ and $H \in X_2^*$ there exists a solution $\omega \in L^\infty(Q_T)$, $u \in D(L)$, $\mathbf{p} \in X_2$ of (11)–(13).*

The main idea of the proof of the above theorem is to apply the method of successive approximation (see [2, 6]). In fact, one defines approximating sequences (ω_k), (u_k), $(\mathbf{p}_k)$ as follows: ω_k is obtained as a solution of (11) by using the previous approximating u_{k-1}, further, u_k is given by (12) by using ω_{k-1} and $\mathbf{p}_{k-1}$, finally one obtains $\mathbf{p}_k$ as a solution of (13) by using the previous approximating functions ω_{k-1} and u_{k-1}.

As in Section 3 it is not difficult to show existence of weak solutions in $(0, \infty)$ by supposing the Volterra property (see [3, 6]). Let $X_i^\infty := L^{p_i}_{\mathrm{loc}}(0, \infty; V_i)$ $(i = 1, 2)$.

Theorem 4.2. *Assume that functions a_i, b_i, f have the Volterra property, further, the conditions of Theorem 4.1 are satisfied for all $0 < T < \infty$. Then for every $G \in L^{q_1}_{\mathrm{loc}}(0, \infty, V^*)$, $H \in L^{q_2}_{\mathrm{loc}}(0, \infty, V_2^*)$ there exist $\omega \in L^\infty(Q_\infty)$, $u \in X_1^\infty$, $\mathbf{p} \in X_2^\infty$ such that their restriction to $(0, T)$ satisfies (11)–(13) for every $0 < T < \infty$.*

One may obtain results also on the long-time behaviour of solutions (see [3, 6]). First, by posing extra coercivity assumptions on functions a_i, b_i (analogously to condition (A4*) in Section 3) boundedness of weak solutions follows.

Theorem 4.3. *Assume that the conditions of Theorem 4.2 are satisfied with further assumptions on coercivity (see [3, 6]) and let $G \in L^\infty(0, \infty; V_1^*)$, $H \in L^\infty(0, \infty; V_2^*)$. Then for the solutions $\omega, u, \mathbf{p}$ formulated in Theorem 4.2, $\omega \in L^\infty(Q_\infty)$, $u \in L^\infty(0, \infty; L^2(\Omega))$, $\mathbf{p} \in L^\infty(0, \infty; V_2)$ hold.*

In case $p_1 = p_2 = p$ and $X_1^\infty = X_2^\infty = X^\infty$ we have results on stabilization of solutions as $t \to \infty$. By assuming the stabilization of functions a_i, b_i, f (analogously to (A6), (B6) in Section 3) we may introduce operators $A_\infty, B_\infty \colon L^\infty(\Omega) \times V \times V \to V^*$ as follows

$$\langle A_\infty(\omega, u, \mathbf{p}), v \rangle := \int_\Omega \left(\sum_{i=1}^n a_{i,\infty}(\omega, u, Du, \mathbf{p}, D\mathbf{p}) D_i v + a_{0,\infty}(\omega, u, Du, \mathbf{p}, D\mathbf{p}) v \right) dx,$$

$$\langle B_\infty(\omega, u, \mathbf{p}), v \rangle := \int_\Omega \left(\sum_{i=1}^n b_{i,\infty}(\omega, u, \mathbf{p}, D\mathbf{p}) D_i v + b_{0,\infty}(\omega, u, \mathbf{p}, D\mathbf{p}) v \right) dx.$$

By adding some extra conditions on uniformy monotonicity (similary to (A7)), further, by assuming that the steady-state of equation (11) is exponentially stable, one may verify

Theorem 4.4. *Assume that the conditions of Theorem 4.3 are satisfied with some further assumptions (see [3]), in addition,* $\lim_{t \to \infty} \|G(t) - G_\infty\|_{V^*} = 0$, $\lim_{t \to \infty} \|H(t) - H_\infty\|_{V^*} = 0$ *hold for some* $G_\infty, H_\infty \in V^*$. *Then there exist unique* $u_\infty, \mathbf{p}_\infty \in V$ *such that* $A_\infty(\omega^*, u_\infty, \mathbf{p}_\infty) = G_\infty$, $B_\infty(\omega^*, u_\infty, \mathbf{p}_\infty) = H_\infty$ *where* ω^* *is the steady-state of* (11). *Further, if* $\omega, u, \mathbf{p}$ *are solutions formulated in Theorem 4.2 then* $\omega(t, \cdot) \to \omega^*$ *in* $L^\infty(\Omega)$, $u(t) \to u_\infty$ *in* $L^2(\Omega)$, $\int_{t-1}^{t+1} \|u(s) - u_\infty\|_V^p \, ds \to 0$, $\int_{t-1}^{t+1} \|\mathbf{p}(s) - \mathbf{p}_\infty\|_V^p \, ds \to 0$ *as* $t \to \infty$.

Similarly to Section 3 one may obtain estimates on the convergences stated in the above theorem. For some examples, see [2, 3, 5, 6].

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