ANNALES UNIV. SCI. BUDAPEST., **47** (2004), 143–160

ON SYSTEMS OF NONLINEAR PARABOLIC FUNCTIONAL DIFFERENTIAL EQUATIONS

By

ÁDÁM BESENYEI

(Received: ???)

1. Introduction

Systems of second order quasilinear parabolic differential equations where also the main part contains functional dependence on the unknown functions were studied e.g. in [1] by L. Simon. There the following equation was considered:

$$D_t u(t,x) \quad \sum_{i=1}^{n} D_i [a_i(t,x,u(t,x),Du(t,x);u)] + a_0(t,x,u(t,x),Du(t,x);u) =$$

$$= f(t,x) \qquad (t,x) \in Q_T = (0,T) \times \Omega, \quad a_i \colon Q_T \times \mathbb{R}^{n+1} \times L^p(0,T;V) \to \mathbb{R},$$

where V denotes a closed linear subset of the Sobolev-space $W^{1,p}(\Omega)$ $(2 \leq \leq p < \infty)$.

Let us now consider a system of this type of equations:

(1)

$$D_t u^{(l)}(t,x) \quad \sum_{i=1}^{n} D_i \left[a_i^{(l)} \left(t,x,u^{(1)}(t,x),\dots,u^{(N)}(t,x), \right.\right.$$

$$\left.\left. Du^{(1)}(t,x),\dots,Du^{(N)}(t,x);u^{(1)},\dots,u^{(N)} \right) \right] +$$

$$+ a_0^{(l)} \left(t,x,u^{(1)}(t,x),\dots,u^{(N)}(t,x), \right.$$

$$\left. Du^{(1)}(t,x),\dots,Du^{(N)}(t,x);u^{(1)},\dots,u^{(N)} \right) =$$

$$= F^{(l)}(t,x), \qquad (t,x) \in Q_T = (0,T) \times \Omega, \quad \Omega \subset \mathbb{R}^n, \quad l = 1,\dots,N.$$

In the next section we define the weak form of the above system and formulate conditions on the coefficients. With these we can prove existence of weak solutions. The conditions are generalizations of the classical Léray–Lions conditions for systems with some special conditions for these type of systems. Finally we show some examples.

2. Existence of weak solutions

First we introduce some notations. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the C^1 regularity property and $2 \le p < \infty$ be a real number. Denote by $W^{1,p}(\Omega)$ the usual Sobolev space of real valued functions with the norm

$$\|u\| = \left(\int_\Omega (|Du|^p + |u|^p)\right)^{\frac{1}{p}}.$$

Let $V_l \subset W^{1,p}(\Omega)$ $(l = 1, \dots, N)$ be a closed linear subspace (e.g. $W_0^{1,p}(\Omega)$ or $W^{1,p}(\Omega)$) and let $V = V_1 \times \cdots \times V_N$. Denote by $L^p(0,T;V)$ the Banach space of measurable functions $u\colon (0,T) \to V$ such that $\|u\|^p$ is integrable and define the norm by

$$\|u\|_{L^p(0,T;V)} = \int_0^T \|u(t)\|_V^p dt.$$

The dual space of $L^p(0,T;V)$ is $L^q(0,T;V^*)$ where $\frac{1}{p} + \frac{1}{q} = 1$ and V^* is the dual space of V. Let $X = L^p(0,T;V)$ and $Y = L^p(0,T;(L^p(\Omega))^N)$. For $u \in$ $\in X$ we shall write $u = (u^{(1)}, \dots, u^{(N)})$, where $u^{(l)} \in L^p(0,T;V_l)$. A vector $\xi \in \mathbb{R}^{(n+1)N}$ is written in the form $\xi = (\zeta_0, \zeta)$, where $\zeta_0 = (\zeta_0^{(1)}, \dots, \zeta_0^{(N)}) \in$ $\in \mathbb{R}^N$ and $\zeta = (\zeta^{(1)}, \dots, \zeta^{(N)}) \in \mathbb{R}^{nN}$. Here $\zeta^{(l)} = (\zeta_1^{(l)}, \dots, \zeta_n^{(l)}) \in \mathbb{R}^n$. Now we formulate 5 essential assumptions on functions $a_i^{(l)}$ $(i = 0, \dots, n;$ $l = 1, \dots, N)$, which (as we will see) are sufficient for existence of weak solutions.

F1. Suppose that $a_i^{(l)}\colon Q_T \times \mathbb{R}^{(n+1)N} \times L^p(0,T;V) \to \mathbb{R}$ are Carathéodory functions for each $v \in L^p(0,T;V)$. This means that they are measurable in (t,x) for every $(\zeta_0,\zeta) \in \mathbb{R}^{(n+1)N}$, and continuous in (ζ_0,ζ) for almost every $(t,x) \in Q_T$ $(i = 0, \dots, n;\ l = 1, \dots, N)$.

F2. Suppose that there exist bounded operators $g_1\colon L^p(0,T;V) \to \mathbb{R}^+$ and $k_1\colon L^p(0,T;V) \to L^q(Q_T)$ such that

$$|a_i^{(l)}(t,x,\zeta_0,\zeta;v)| \le g_1(v)\left(|\zeta_0|^{p\ 1} + |\zeta|^{p\ 1}\right) + \left[k_1(v)\right](t,x)$$

for a.e. $(t,x) \in Q_T$, each $(\zeta_0,\zeta) \in \mathbb{R}^{(n+1)N}$ and $v \in L^p(0,T;V)$ $(i = 0,\dots,n;\ l = 1,\dots,N)$.

F3. Suppose that for each $\zeta \ne \eta \in \mathbb{R}^{nN}$, a.e. $(t,x) \in Q_T$, each $\zeta_0 \in \mathbb{R}^N$ and each $v \in L^p(0,T;V)$

$$\sum_{l=1}^{N}\sum_{i=1}^{n}\left(a_i^{(l)}(t,x,\zeta_0,\zeta;v) \quad a_i^{(l)}(t,x,\zeta_0,\eta;v)\right)(\zeta_i^{(l)} \quad \eta_i^{(l)}) > 0.$$

F4. Suppose that there exist operators $g_2\colon L^p(0,T;V) \to \mathbb{R}^+$ and $k_2\colon L^p(0,T;V) \to L^1(Q_T)$ such that

$$\sum_{l=1}^{N}\sum_{i=0}^{n} a_i^{(l)}(t,x,\zeta_0,\zeta;v)\zeta_i^{(l)} \ge g_2(v)\ \ |\zeta_0|^p + |\zeta|^p) \quad [k_2(v)](t,x)$$

for a.e. $(t,x) \in Q_T$, each $(\zeta_0,\zeta) \in \mathbb{R}^{(n+1)N}$ and $v \in L^p(0,T;V)$ $(i = = 0,\dots,n;\ l = 1,\dots,N)$. Further, operators g_2, k_2 has the following property:

$$\lim_{\|v\|_{L^p(0,T;V)}\to\infty}\left(g_2(v)\|v\|^{p\ 1}_{L^p(0,T;V)} \quad \frac{\|k_2(v)\|_{L^1(Q_T)}}{\|v\|_{L^p(0,T;V)}}\right) = +\infty.$$

F5. Suppose that if $u_k \to u$ weakly in $L^p(0,T;V)$ and strongly in $L^p(0,T;(L^p(\Omega))^N)$, then

$$\lim_{k\to\infty}\|a_i^{(l)}(\cdot\,,u_k(\cdot),Du_k(\cdot);u_k) \quad a_i^{(l)}(\cdot\,,u_k(\cdot),Du_k(\cdot);u)\|_{L^q(Q_T)} = 0.$$

$(i = 0,\dots,n;\ l = 1,\dots,N)$

We now define the weak form of system (1). Let us introduce first the operator $A\colon L^p(0,T;V) \to L^q(0,T;V^*)$. For $u = (u^{(1)},\dots,u^{(N)}) \in \in L^p(0,T;V)$ and $v = (v^{(1)},\dots,v^{(N)}) \in L^p(0,T;V)$ define

$$[A(u),v] :=$$

$$\sum_{l=1}^{N}\int_{Q_T}\left[\sum_{i=1}^{n} a_i^{(l)}\left(t,x,u(t,x),Du(t,x);u\right)D_i v^{(l)} +\right.$$

$$+ a_0^l(t,x,u(t,x),Du(t,x);u)\Big) v^{(l)}\Bigg] dtdx,$$

where D_i denotes the operator of (distributional) partial differentiating with respect to x_i and $D = (D_1, \dots, D_N)$. As usual let $L\colon L^p(0,T;V) \to L^q(0,T;V^*)$ be the following operator:

$$D(L) = \{u \in X\colon D_t u \in X^*, u(0) = 0\}, \qquad Lu = D_t u.$$

With operator A we define the weak form of system (1) by

$$D_t u + A(u) = F.$$

In the next theorem we prove some important properties of A from which existence of weak solution follows.

THEOREM 1. *Assume that conditions F1–F5 are fulfilled. Then* $A\colon X \to X^*$ *is bounded, demicontinuous, coercive and pseudomonotone with respect to* $D(L)$.

PROOF. The proof is based on elementary techincs and on Hölder's inequality.

BOUNDEDNESS. From triangle inequality it is clear that it is sufficient to deal with only one integral in $[A(u), v]$. This can be estimated by Hölder's inequality:

$$(2) \qquad \left| \int_{Q_T} a_i^{(l)}(t,x,u(t,x),Du(t,x);u) D_i v^{(l)}(t,x)dtdx \right| \le$$

$$\le \left(\int_{Q_T} |a_i^{(l)}(t,x,u(t,x),Du(t,x);u)|^q dtdx \right)^{\frac{1}{q}} \left(\int_{Q_T} |D_i v^{(l)}(t,x)|^p dtdx \right)^{\frac{1}{p}}.$$

(In case $i = 0$ we replace $D_i v^{(l)}$ by $v^{(l)}$.) On the right hand side of (2) the second term is less or equal than $\|v\|_X$ and the first term can be estimated by the inequality $|a+b|^r \le 2^{r-1} \cdot (|a|^r + |b|^r)$:

$$(3) \quad \left(\int_{Q_T} |a_i^{(l)}(t,x,u(t,x),Du(t,x);u)|^q dtdx \right)^{\frac{1}{q}} \le$$

$$\le const \cdot \left(\int_{Q_T} \left[g_1(u)^q \left(|u(t,x)|^{(p-1)q} + |Du(t,x)|^{(p-1)q} \right) + \right.\right.$$

$$+ |[k_1(u)](t,x)|^q \Big] \, dt\,dx \Bigg)^{\frac{1}{q}} \leq$$

$$\leq const \cdot \left[g_1(u) \left(\int_{Q_T} |u|^p + |Du|^p \right)^{\frac{1}{q}} + \left(\int_{Q_T} |k_1(u)|^q \right)^{\frac{1}{q}} \right] =$$

$$= const \cdot \left(g_1(u) \|u\|_X^{\frac{p}{q}} + \|k_1(u)\|_{L^q(Q_T)} \right).$$

Summing the above estimations with respect to i and l we get:

$$|[A(u), v]| \leq const \cdot \left(g_1(u) \|u\|_X^{\frac{p}{q}} + \|k_1(u)\|_{L^q(Q_T)} \right) \|v\|_X.$$

This means that $\|A(u)\|_{X^*} \leq const \cdot \left(g_1(u) \|u\|_X^{\frac{p}{q}} + \|k_1(u)\|_{L^q(Q_T)} \right)$. From here by boundedness of operators g_1 and k_1 follows the boundedness of A.

DEMICONTINUITY. Assume that $u_k \to u$ strongly in X. Then there exists a subsequence $(\tilde{u}_k) \subset (u_k)$, such that $(\tilde{u}_k) \to u$ and $(D\tilde{u}_k) \to Du$ for a.e. $(t,x) \in Q_T$. We show that for each $v \in X$ we have $[A(\tilde{u}_k) \quad A(u), v] \to 0$, then using the subsequence trick the proof of demicontinuity will be finished. It is useful to introduce operator $\tilde{A}_u \colon X \to X^*$ (u is fixed) defined by

$$[\tilde{A}_u(v), w] := \sum_{l=1}^{N} \int_{Q_T} \left[\sum_{i=1}^{n} a_i^{(l)}(t,x,v(t,x),Dv(t,x);u) \, D_i w^{(l)}(t,x) + \right.$$

$$\left. + a_0^{(l)}(t,x,v(t,x),Dv(t,x);u) \, w^{(l)}(t,x) \right] dt\,dx.$$

We prove that $A(\tilde{u}_k) \quad \tilde{A}_u(\tilde{u}_k) \to 0$ and $\tilde{A}_u(\tilde{u}_k) \quad A(u) \to 0$ weakly in X^*. It is easy to see (from triangle and Hölder's inequality) that it is sufficient to show

(4) $\quad \|a_i^{(l)}(\cdot\,, \tilde{u}_k(\cdot), D\tilde{u}_k(\cdot); \tilde{u}_k) \quad a_i^{(l)}(\cdot\,, \tilde{u}_k(\cdot), D\tilde{u}_k(\cdot); u)\|_{L^q(Q_T)} \to 0$

and

(5) $\quad \|a_i^{(l)}(\cdot\,, \tilde{u}_k(\cdot), D\tilde{u}_k(\cdot); u) \quad a_i^{(l)}(\cdot\,, u(\cdot), Du(\cdot); u)\|_{L^q(Q_T)} \to 0.$

The strong convergence in X implies the weak convergence in X, and because of the continuous imbedding $X \to Y$ it implies the weak convergence in Y, too. So that from F5 it follows that (4) is true indeed. On the other hand, from condition F1 we know that $a_i^{(l)}$ is continuous in (ζ_0, ζ), hence

$$a_i^{(l)}(t,x,\tilde{u}_k(t,x), D\tilde{u}_k(t,x);u) \to a_i^{(l)}(t,x,u(t,x),Du(t,x);u)$$

for a.e. $(t,x) \in Q_T$, by the almost everywhere convergence of $\tilde{u}_k$ and $D\tilde{u}_k$ in Q_T. Further,

$$|a_i^{(l)}(t,x,\tilde{u}_k(t,x), D\tilde{u}_k(t,x);u)|^q \le$$
$$\le g_1(u)^q \big(|\tilde{u}_k(t,x)|^p + |D\tilde{u}_k(t,x)|^p\big) + |[k_1(u)](t,x)|^q = f_k(t,x).$$

Since $(\tilde{u}_k)$ is convergent in X, (f_k) is convergent in $L^1(Q_T)$, consequently equiintegrable in $L^1(Q_T)$, too. Hence functions $a_i^{(l)}(\cdot\,,\tilde{u}_k(\cdot), D\tilde{u}_k(\cdot);u)$ $(k \in \in \mathbb{N})$ are equiintegrable in $L^q(Q_T)$. Then by Vitali's theorem we have

$$\lim_{k\to\infty} \|a_i^{(l)}(\cdot\,,\tilde{u}_k(\cdot), D\tilde{u}_k(\cdot);u) - a_i^{(l)}(\cdot\,,u(\cdot),Du(\cdot);u)\|_{L^q(Q_T)} = 0.$$

REMARK. Observe that we have shown also the following facts: $A(\tilde{u}_k) - \tilde{A}_u(\tilde{u}_k) \to 0$ weakly in X^* and $[A(\tilde{u}_k) - \tilde{A}_u(\tilde{u}_k), v_k] \to 0$, if (v_k) is a bounded sequence in X.

COERCITIVITY. From condition F4 we get

$$[A(u),u] \ge \int_{Q_T} \big[g_2(u)|u(t,x)|^p + |Du(t,x)|^p - [k_2(u)](t,x)\big]\,dt\,dx =$$
$$= g_2(u)\|u\|_X^p - \|k_2(u)\|_{L^1(Q_T)},$$

thus using F4 again we obtain

$$\lim_{\|u\|_X\to\infty} \frac{[A(u),u]}{\|u\|_X} \ge \lim_{k\to\infty} \left[g_2(u)\|u\|_X^{p-1} - \frac{\|k_2(u)\|_{L^1(Q_T)}}{\|u\|_X}\right] = +\infty.$$

PSEUDOMONOTONICITY. Let us suppose that

(6) $\quad (u_k) \to u$ weakly in X and $(D_t u_k) \to D_t u$ weakly in X^*,

further

(7) $$\limsup_{k\to\infty} [A(u_k), u_k - u] \le 0.$$

By using the subsequence trick it is sufficient to show that for a subsequence $(\tilde{u}_k) \subset (u_k)$

$$\lim_{k\to\infty} [A(\tilde{u}_k), \tilde{u}_k - u] = 0 \quad \text{and} \quad A(\tilde{u}_k) \to A(u) \text{ weakly in } X^*.$$

Since the imbedding $W^{1,p}(\Omega) \to L^p(\Omega)$ is compact and (u_k) is bounded in X and (Du_k) is bounded in X^* by its weak convergence, hence from the well known imbedding theorem (see [4]) there exists a subsequence $(\tilde{u}_k) \subset (u_k)$ such that $\tilde{u}_k \to u$ in Y. Then by using the above remark we obtain

(8) $$\lim_{k\to\infty} [A(\tilde{u}_k) \quad \tilde{A}_u(\tilde{u}_k), \tilde{u}_k \quad u] = 0.$$

Comparing this with (7) it follows that

(9) $$\limsup_{k\to\infty} [\tilde{A}_u(\tilde{u}_k), \tilde{u}_k \quad u] \leq 0.$$

We know that $\tilde{A}_u$ is pseudomonotone with respect to $D(L)$ (see [2]), hence from conditions (6) and (9) we get

(10) $$\lim_{k\to\infty} [\tilde{A}_u(\tilde{u}_k), \tilde{u}_k \quad u] = 0 \quad \text{and} \quad \tilde{A}_u(\tilde{u}_k) \to \tilde{A}_u(u)(= A(u)) \text{ weakly in } X^*.$$

From this, by using (8) we have $\lim_{k\to\infty} [A(\tilde{u}_k), \tilde{u}_k \quad u] = 0$. On the other hand, we have shown in the proof of demicontinuity that $\tilde{A}_u(\tilde{u}_k) \quad A(\tilde{u}_k) \to 0$ weakly in X^*, so that by using the second part of (10) we obtain $A(\tilde{u}_k) \to$ $\to A(u)$ weakly in X^*. This completes the proof. ∎

COROLLARY 1. *For every $F \in X^*$ the equation*

$$D_t u + A(u) = F, \qquad u(0) = 0$$

has got a solution $u \in D(L)$.

PROOF. Since operator D_t is closed, linear and maximal monotone (see e.g. [5]), therefore the statement follows from the preceding theorem and theorem 4 in [3]. ∎

3. Examples

In this section we deal with a general form of functions $a_i^{(l)}$ which fulfil conditions F1–F5. In the end we show some concrete examples.

3.1. General case

Suppose that function $a_i^{(l)}(t,x,\zeta_0,\zeta;v)$ has the form:

$$(11)\quad a_i^{(l)}(t,x,\zeta_0,\zeta;v) = \left[H^{(l)}(v)\right](t,x)b_i^{(l)}(t,x,\zeta_0,\zeta) + \\ + \left[G^{(l)}(v)\right](t,x)d_i^{(l)}(t,x,\zeta_0,\zeta) \text{ if } i \neq 0, \text{ and}$$

$$(12)\quad a_0^{(l)}(t,x,\zeta_0,\zeta;v) = \left[H^{(l)}(v)\right](t,x)b_0^{(l)}(t,x,\zeta_0,\zeta) + \\ + \left[G_0^{(l)}(v)\right](t,x)d_0^{(l)}(t,x,\zeta_0,\zeta),$$

where $b_i^{(l)}, d_i^{(l)}, H^{(l)}, G^{(l)}, G_0^{(l)}$ have the following properties.

K1. Functions $b_i^{(l)}: Q_T \times \mathbb{R}^{(n+1)N} \to \mathbb{R}$ and $d_i^{(l)}: Q_T \times \mathbb{R}^{(n+1)N} \to \mathbb{R}$ has the Carathéodory property. This means that they are measurable in (t,x) for every $(\zeta_0,\zeta) \in \mathbb{R}^{(n+1)N}$, and continuous in (ζ_0,ζ) for a.e. $(t,x) \in Q_T$ ($i = 1,\dots,n$; $l = 1,\dots,N$).

K2. There exist constants $c_1 > 0$, $0 \le r < p - 1$ and a function $k_1 \in L^q(Q_T)$ such that

a) $|b_i^{(l)}(t,x,\zeta_0,\zeta)| \le c_1(|\zeta_0|^{p-1} + |\zeta|^{p-1}) + k_1(t,x)$,

b) $|d_i^{(l)}(t,x,\zeta_0,\zeta)| \le c_1(|\zeta_0|^r + |\zeta|^r)$

for a.e. $(t,x) \in Q_T$ and each $(\zeta_0,\zeta) \in \mathbb{R}^{(n+1)N}$ ($i = 1,\dots,n$; $l = 1,\dots,N$).

K3. For each $\zeta \neq \eta$

a) $$\sum_{i=1}^n [b_i^{(l)}(t,x,\zeta_0,\zeta) - b_i^{(l)}(t,x,\zeta_0,\eta)](\zeta_i^{(l)} - \eta_i^{(l)}) > 0,$$

b) $$\sum_{i=1}^n [d_i^{(l)}(t,x,\zeta_0,\zeta) - d_i^{(l)}(t,x,\zeta_0,\eta)](\zeta_i^{(l)} - \eta_i^{(l)}) \ge 0$$

for a.e. $(t,x) \in Q_T$ and each $\zeta_0 \in \mathbb{R}^N$ ($l = 1,\dots,N$).

K4. There exist a constant $c_2 > 0$ and a function $k_2 \in L^1(Q_T)$ such that

a) $$\sum_{i=0}^n b_i^{(l)}(t,x,\zeta_0,\zeta)\zeta_i^{(l)} \ge c_2(|\zeta_0^{(l)}|^p + |\zeta^{(l)}|^p) - k_2(t,x),$$

b) $\sum_{i=1}^{n} d_i^{(l)}(t,x,\zeta_0,\zeta)\zeta_i^{(l)} \geq 0$

for a.e. (t,x) and each $(\zeta_0,\zeta) \in \mathbb{R}^{(n+1)N}$ $(l = 1,\dots,N)$.

K5.

a) The operator $H^{(l)}\colon L^p(0,T;(L^p(\Omega))^N) \to L^\infty(Q_T)$ is bounded and continuous such that for every $v \in L^p(0,T;(L^p(\Omega))^N)$ $[H^{(l)}(v)](t,x) \geq c_3 > 0$ holds for a.e. $(t,x) \in Q_T$.

b) The operators $G^{(l)}, G_0^{(l)}\colon L^p(0,T;(L^p(\Omega))^N) \to L^{\frac{p}{p-r-1}}(Q_T)$ are bounded, continuous where r is given in K2/b. Further, for each $v \in L^p(0,T;(L^p(\Omega))^N)$ we have $[G^{(l)}(v)](t,x) \geq 0$ for a.e. $(t,x) \in$ $\in Q_T$ and

$$\lim_{\|v\|_{L^p(0,T;V)}\to\infty} \frac{\int_{Q_T} |G_0^{(l)}(v)(t,x)|^{\frac{p}{p-r-1}}\,dt\,dx}{\|v\|^p_{L^p(0,T;V)}} = 0, \quad l = 1,\dots,N. \tag{13}$$

CLAIM 1. *Assume that conditions K1–K5 hold. Then functions defined in* (11), (12) *satisfy conditions F1–F5.*

For the proof we need a technical lemma.

LEMMA 1. *Let us introduce the following operators:*

$$[H(v)](t,x) = \sum_{l=1}^{N} |[H^{(l)}(v)](t,x)|$$

$$[G(v)](t,x) = \sum_{l=1}^{N} |[G^{(l)}(v)](t,x)|$$

$$[G_0(v)](t,x) = \sum_{l=1}^{N} |[G_0^{(l)}(v)](t,x)|.$$

Then operators H, G and G_0 fulfil the conditions formulated in K5 on $H^{(l)}$, $G^{(l)}$ and $G_0^{(l)}$, respectively.

PROOF OF LEMMA 1. We have to prove only (13) which follows easily by estimating the integrand by $|a+b|^s \leq 2^{s-1}(|a|^s + |b|^s)$. ∎

PROOF OF CLAIM 1.

CONDITION F1. From K1 obviously follows F1.

CONDITION F2. Let $i > 0$ and $r > 0$. It is obvious that

$$|[H^{(l)}(v)](t,x)b_i^{(l)}(t,x,\zeta_0,\zeta)| \leq$$
$$\leq \|H(v)\|_{L^\infty(Q_T)}\left(c_1\left(|\zeta_0|^{p\ \ 1}+|\zeta|^{p\ \ 1}\right)+k_1(t,x)\right).$$

On the other hand by using Young's inequality with conjugate exponents $1 < p_1 = \frac{p\ \ 1}{r} < \infty$ and $q_1 = \frac{p\ \ 1}{p\ \ r\ \ 1}$ we get

$$(14)\quad |[G^{(l)}(v)](t,x)d_i^{(l)}(t,x,\zeta_0,\zeta)| \leq |[G(v)](t,x)d_i^{(l)}(t,x,\zeta_0,\zeta)|$$
$$\leq \frac{|d_i^{(l)}(t,x,\zeta_0,\zeta)|^{p_1}}{p_1}+\frac{|[G(v)](t,x)|^{q_1}}{q_1}.$$

Estimating by K2/b and $|a+b|^s \leq 2^{s\ \ 1}(|a|^s+|b|^s)$ we obtain

$$|[G^{(l)}(v)](t,x)d_i^{(l)}(t,x,\zeta_0,\zeta)| \leq const\cdot\ \ |\zeta_0|^{rp_1}+|\zeta|^{rp_1}+|[G(v)](t,x)|^{q_1})$$
$$(15)\qquad = const\cdot\left(|\zeta_0|^{p\ \ 1}+|\zeta|^{p\ \ 1}+|[G(v)](t,x)|^{q_1}\right).$$

Combining the above estimations we have

$$|a_i^{(l)}(t,x,\zeta_0,\zeta;v)| \leq const\cdot\Big[\left(\|H(v)\|_{L^\infty(Q_T)}+1\right)\left(|\zeta_0|^{p\ \ 1}+|\zeta|^{p\ \ 1}\right)+$$
$$+\|H(v)\|_{L^\infty(Q_T)}k_1(t,x)+|[G(v)](t,x)|^{q_1}\Big].$$

By the boundedness of operator H and by the continuous imbedding $X \to Y$ we have that $\|H(\cdot)\|_{L^\infty(Q_T)}$ is a bounded $X \to \mathbb{R}^+$ functional. Further, from $k_1 \in L^q(Q_T)$ it follows that $\|H(\cdot)\|_{L^\infty(Q_T)}k_1$ is a bounded $X \to L^q(Q_T)$ operator. Observe that $q_1 q = \frac{p}{p\ \ r\ \ 1}$ so that

$$(16)\qquad \int_{Q_T}\ |[G(v)](t,x)|^{q_1})^q\,dt\,dx = \int_{Q_T}|[G(v)](t,x)|^{\frac{p}{p\ \ r\ \ 1}}dt\,dx$$
$$= \left(\|G(v)\|_{L^{\frac{p}{p\ \ r\ \ 1}}(Q_T)}\right)^{\frac{p}{p\ \ r\ \ 1}}.$$

Due to boundedness of G this means that $|G(\cdot)|^{q_1}$ is a bounded $X \to L^q(Q_T)$ operator.

Now let $r = 0$. Observe that $q_1 = 1$, moreover from K2/b we have $|d_i^{(l)}(t, x, \zeta_0, \zeta)| \leq 2c_1$. So in this case we also have an inequality similar to (15):

$$|[G^{(l)}(v)](t,x)d_i^{(l)}(t,x,\zeta_0,\zeta)| \leq const \cdot |[G(v)](t,x)|^{q_1}.$$

This means that this case can be treated in the same way. This completes the proof in case $i > 0$. Case $i = 0$ is the same, we only have to replace G by G_0.

CONDITION F3. Using condition K3 and K5/a we get for $\zeta \neq \eta$

$$\sum_{l=1}^{N}\sum_{i=1}^{n}\left(a_i^{(l)}(t,x,\zeta_0,\zeta;v) \quad a_i^{(l)}(t,x,\zeta_0,\eta;v)\right)(\zeta_i^{(l)} \quad \eta_i^{(l)}) =$$

$$= \sum_{l=1}^{N}[H^{(l)}(v)](t,x)\sum_{i=1}^{n}\left(b_i^{(l)}(t,x,\zeta_0,\zeta) \quad b_i^{(l)}(t,x,\zeta_0,\eta)\right)(\zeta_i^{(l)} \quad \eta_i^{(l)})+$$

$$+\sum_{l=1}^{N}[G^{(l)}(v)](t,x)\sum_{i=1}^{n}\left(d_i^{(l)}(t,x,\zeta_0,\zeta) \quad d_i^{(l)}(t,x,\zeta_0,\eta)\right)(\zeta_i^{(l)} \quad \eta_i^{(l)}) > 0.$$

CONDITION F4. Taking into account conditions K4 and K5 we obtain

(17) $$\sum_{l=1}^{N}\sum_{i=0}^{n}a_i^{(l)}(t,x,\zeta_0,\zeta;v)\zeta_i^{(l)} = \sum_{l=1}^{N}\left[H^{(l)}(v)\right](t,x)\cdot$$

$$\cdot\sum_{i=0}^{n}b_i^{(l)}(t,x,\zeta_0,\zeta)\zeta_i^{(l)} + \sum_{l=1}^{N}\left[G^{(l)}(v)\right](t,x)\sum_{i=1}^{n}d_i^{(l)}(t,x,\zeta_0,\zeta)\zeta_i^{(l)}+$$

$$+\sum_{l=1}^{N}\left[G_0^{(l)}(v)\right](t,x)d_0^{(l)}(t,x,\zeta_0,\zeta)\zeta_i^{(l)} \geq$$

$$\geq \sum_{l=1}^{N}c_3c_2\left(\left|\zeta_0^{(l)}\right|^p + \left|\zeta^{(l)}\right|^p\right) \quad c_3k_2(t,x)+$$

$$+\sum_{l=1}^{N}\left[G_0^{(l)}(v)\right](t,x)d_0^{(l)}(t,x,\zeta_0,\zeta)\zeta_i^{(l)} \geq$$

$$\geq c_4c_3c_2 \quad |\zeta_0|^p + |\zeta|^p) \quad c_3Nk_2(t,x)+$$

$$+\sum_{l=1}^{N}\left[G_0^{(l)}(v)\right](t,x)d_0^{(l)}(t,x,\zeta_0,\zeta)\zeta_i^{(l)}.$$

In the last estimation we used inequality $|a+b|^s \le 2^{s-1}(|a|^s+|b|^s)$. Put $c' = c_4c_3c_2$ and investigate only the terms in the last sum. Let $\varepsilon > 0$ be fixed a constant such that $\frac{\varepsilon^p}{p} < \frac{c'}{3N}$, and use the ε-inequality with exponents $p,\ q$. Then we have

$$(18)\qquad |[G_0^{(l)}(v)](t,x)d_0^{(l)}(t,x,\zeta_0,\zeta)\zeta_i^{(l)}| \le$$
$$\le |[G_0(v)](t,x)d_0^{(l)}(t,x,\zeta_0,\zeta)\zeta_i^{(l)}| \le$$
$$\le \frac{\varepsilon^p}{p}|\zeta_i^{(l)}|^p + \frac{\varepsilon^{-q}}{q}|[G_0(v)](t,x)d_0^{(l)}(t,x,\zeta_0,\zeta)|^q.$$

The first term in the right hand side of (18) is less or equal than $\frac{c'}{3N}\left(|\zeta_0|^p + |\zeta|^p\right)$. In the second term using the ε-inequality with $\mu > 0$ (defined later) and exponents $p_1,\ q_1$ similarly to (14), (15), the following estimation holds:

$$(19)\quad \left|\left[G_0^{(l)}(v)\right](t,x)d_i^{(l)}(t,x,\zeta_0,\zeta)\right|^q \le$$
$$\le const\cdot\left(\mu^{p_1}\left(|\zeta_0|^{p-1}+|\zeta|^{p-1}\right)+\mu^{-q_1}|[G_0(v)](t,x)|^{q_1}\right)^q \le$$
$$\le c^*\mu^{p_1q}\left(|\zeta_0|^p+|\zeta|^p\right)+c^*\mu^{-q_1q}|[G_0(v)](t,x)|^{q_1q}.$$

Let μ be such that $\frac{c^*\mu^{p_1q}\varepsilon^{-q}}{q} < \frac{c'}{3N}$. Then substituting (18) and (19) into (17)

$$\sum_{l=1}^{N}\sum_{i=0}^{n}a_i^{(l)}(t,x,\zeta_0,\zeta;v)\zeta_i^{(l)} \ge$$
$$\ge \frac{c'}{3}\left(|\zeta_0|^p+|\zeta|^p\right) - \underbrace{\left(c_3Nk_2(t,x)+Nd^*|[G_0(v)](t,x)|^{q_1q}\right)}_{=:[h(v)](t,x)}$$

where $h(v)\in L^1(Q_T)$ following from (16) (and $k_2 \in L^1(Q_T)$). Moreover

$$\|h(v)\|_{L^1(Q_T)} \le c_3N\|k_2\|_{L^1(Q_T)} + Nd^*\int_{Q_T}|[G_0(v)](t,x)|^{\frac{p}{p-r-1}}dtdx.$$

From the lemma we know that G_0 fulfil (13), hence

$$\lim_{\|v\|_X\to\infty}\|v\|_X^{p-1}\left(\frac{c'}{3} - \frac{\|h(v)\|_{L^1(Q_T)}}{\|v\|_X^p}\right) = \lim_{\|v\|_X\to\infty}\frac{c'}{3}\|v\|_X^{p-1} = +\infty.$$

CONDITION F5. Let $r > 0$. Suppose that $u_k \to u$ weakly in X and strongly in Y. Then (u_k) is bounded in X. Therefore from K2/a follows that $b_i^{(l)}(\cdot\,, u_k(\cdot), Du_k(\cdot))$ $(k \in \mathbb{N})$ is bounded in $L^q(Q_T)$, since it is easy to see (similarly to (3)) that

$$\int_{Q_T} |b_i^{(l)}(t,x,u_k(t,x), Du_k(t,x)|^q \, dt dx \le$$
$$\le const \cdot \int_{Q_T} \left[|u_k(t,x)|^{(p-1)q} + |Du_k(t,x)|^{(p-1)q} + |k_1(t,x)|^q \right] dt dx \le$$
$$\le const \cdot (\|u_k\|_X^p + \|k_1\|_{L^q(Q_T)}^q) \le K.$$

Further observe that $d_i^{(l)}(\cdot\,, u_k(\cdot), Du_k(\cdot))$ is bounded in $L^{\frac{p}{r}}(Q_T)$, since by K2/b

$$\int_{Q_T} |d_i^{(l)}(t,x,u_k(t,x), Du_k(t,x))|^{\frac{p}{r}} \, dt dx \le$$
$$\le \int_{Q_T} \left[|u_k(t,x)|^{r\frac{p}{r}} + |Du_k(t,x)|^{r\frac{p}{r}} \right] dt dx = \|u_k\|_X^p \le K.$$

Hence

$$\int_{Q_T} |([H^{(l)}(u_k)](t,x) - [H^{(l)}(u)](t,x)) b_i^{(l)}(t,x,u_k(t,x), Du_k(t,x))|^q \, dt dx \le$$
$$\le \|H^{(l)}(u_k) - H^{(l)}(u)\|_{L^\infty(Q_T)}^q \int_{Q_T} |b_i^{(l)}(t,x,u_k(t,x), Du_k(t,x))|^q \, dt dx \le$$
$$\le K \|H^{(l)}(u_k) - H^{(l)}(u)\|_{L^\infty(Q_T)} \to 0,$$

by using the continuity of $H^{(l)}$. On the other hand, Hölder's inequality with exponents p_1, q_1 shows that

$$\int_{Q_T} |([G^{(l)}(u_k)](t,x) - [G^{(l)}(u)](t,x)) d_i^{(l)}(t,x,u_k(t,x), Du_k(t,x))|^q \, dt dx \le$$
$$\le \left(\int_{Q_T} |d_i^{(l)}(t,x,u_k(t,x), Du_k(t,x)|^{\frac{p}{p-1}\frac{p-1}{r}} \, dt dx \right)^{\frac{1}{p_1}} \cdot$$
$$\cdot \left(\int_{Q_T} |[G^{(l)}(u_k)](t,x) - [G^{(l)}(u)](t,x)|^{\frac{p}{p-1}\frac{p-1}{p-r-1}} \, dt dx \right)^{\frac{1}{q_1}} \le$$

$$\leq K^{\frac{1}{p_1}} \| G^{(l)}(u_k) \quad G^{(l)}(u) \|_{L^{\frac{p}{p \ r \ 1}}(Q_T)}^{\frac{p \ r \ 1}{p}} \to 0,$$

since $G^{(l)}$ is continuous. This means that

$$\begin{aligned}(20) \quad \| a_i^{(l)}(\cdot\,, u_k(\cdot), Du_k(\cdot); u_k) \quad a_i^{(l)}(\cdot\,, u_k(\cdot), Du_k(\cdot); u) \|_{L^q(Q_T)} \leq \\ \leq \| (H^{(l)}(u_k) \quad H^{(l)}(u)) b_i^{(l)}(\cdot\,, u_k(\cdot), Du_k(\cdot)) \|_{L^q(Q_T)} + \\ + \| (G^{(l)}(u_k) \quad G^{(l)}(u)) d_i^{(l)}(\cdot\,, u_k(\cdot), Du_k(\cdot)) \|_{L^q(Q_T)} \to 0.\end{aligned}$$

If $r = 0$, then the first term on the right hand side of (20) tends to 0. Since $\frac{p}{p \ r \ 1} = q$ (hence G maps to $L^q(Q_T)$ continuously) and $|b_i^{(l)}(t, x, \zeta_0, \zeta)| \leq$ $\leq 2c_1$, so that

$$\begin{aligned}\| (G^{(l)}(u_k) \quad G^{(l)}(u)) d_i^{(l)}(\cdot\,, u_k(\cdot), Du_k(\cdot)) \|_{L^q(Q_T)} \leq \\ \leq 2c_1 \| (G^{(l)}(u_k) \quad G^{(l)}(u)) \|_{L^q(Q_T)} \to 0.\end{aligned}$$

Hence the second term in the right hand side of (20) tends to 0, too. Case $i = 0$ can be treated similarly, replacing $G^{(l)}$ by $G_0^{(l)}$.

3.2. Concrete examples

3.2.1. Operator $H^{(l)}$

Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\phi \geq c > 0$. Let us introduce the following operators on $L^p(0, T; (L^p(\Omega))^N)$:

$$[\tilde{H}_1(v)](t, x) := \phi \left(\int_{Q_t} \sum_{j=1}^{N} b_j v^{(j)} \right), \quad \text{where } \ b_j \in L^q(Q_T) \ \ (1 \leq j \leq N),$$

$$[\tilde{H}_2(v)](t, x) := \phi \left(\left[\int_{Q_t} |v|^\alpha \right]^{\frac{1}{\alpha}} \right), \quad \text{where } \ 1 \leq \alpha \leq p.$$

CLAIM 2. *The above $\tilde{H}_1$ and $\tilde{H}_2$ fulfil condition K5/a.*

PROOF. We prove only the case of $\tilde{H}_1$, the other can be made by similar techincs. From Hölder's inequality we know that $b_j v^{(j)} \in L^1(Q_T)$, so that

$\tilde{H}_1$ is well defined, and obviously $\tilde{H}_1(v) \geq c > 0$. On the other hand, if $\|v\|_Y \leq K$ then we have

$$\left| \int_{Q_t} \sum_{j=1}^{N} b_j v^{(j)} \right| \leq \sum_{j=1}^{N} \int_{Q_T} |b_j v^{(j)}| \leq K \sum_{j=1}^{N} \|b_j\|_{L^q(Q_T)},$$

from where by continuity of ϕ follows that $\tilde{H}_1$ maps to $L^\infty(Q_T)$ and it is bounded indeed. Further, if $(v_k) \to v$ in $L^p(0, T; (L^p(\Omega))^N)$ then we have

$$\left| \int_{Q_t} \sum_{j=1}^{N} b_j v_k^{(j)} \quad \int_{Q_t} \sum_{j=1}^{N} b_j v^{(j)} \right| \leq$$

$$\leq \sum_{j=1}^{N} \left(\int_{Q_T} |b_j|^q \right)^{\frac{1}{q}} \left(\int_{Q_T} |v_k^{(j)} \quad v^{(j)}|^p \right)^{\frac{1}{p}} \to 0,$$

therefore by continuity of ϕ it follows that $\tilde{H}_1(v_k) \to \tilde{H}_1(v)$ in $L^\infty(Q_T)$. This completes the proof of continuity. ∎

3.2.2. Operators $G^{(l)}$, $G_0^{(l)}$

Let $\psi : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $|\psi(y)| \leq const \cdot$ $\cdot |y|^{p \quad r_0 \quad 1}$ holds for some $0 \leq r_0 < p \quad 1$. Let us introduce the following operators on $L^p(0, T; (L^p(\Omega))^N)$:

$$[\tilde{G}_1(v)](t, x) := \psi \left(\int_0^t \sum_{j=1}^{N} a_j(\tau, x) v^{(j)}(\tau, x) d\tau \right),$$

$$[\tilde{G}_2(v)](t, x) := \psi \left(\int_\Omega \sum_{j=1}^{N} a_j(t, \xi) v^{(j)}(t, \xi) d\xi \right),$$

where $a_j \in L^\infty(Q_T)$ $(1 \leq j \leq N)$,

$$[\tilde{G}_3(v)](t, x) := \psi \left(\left[\int_0^t |v(\tau, x)|^\alpha d\tau \right]^{\frac{1}{\alpha}} \right), \text{ where } 1 \leq \alpha \leq p.$$

CLAIM 3. *The above $\tilde{G}_i$ fulfil conditions made on $G_0^{(l)}$ in K5/b with $0 \leq r < r_0$. (If $\psi \geq 0$, then obviously the nonnegativity condition is fulfilled, too.)*

PROOF. We show only the case of operator $\tilde{G}_1$. Let be $0 \le r < r_0 < p - 1$ then from properties of ψ it is obvious that

$$\int_{Q_T} |[\tilde{G}_1(v)](t,x)|^{\frac{p}{p-r-1}} dt dx \le$$

$$\le const \cdot \int_{Q_T} \left(\sum_{j=1}^{N} \int_0^T \|a_j\|_{L^\infty(Q_T)} |v^{(j)}(\tau,x)| d\tau \right)^{p\lambda} dt dx \le$$

$$\le const \cdot \int_{Q_T} \left(\cdot \sum_{j=1}^{N} \int_0^T |v(\tau,x)| d\tau \right)^{p\lambda} dt dx =$$

$$= const \cdot \int_{Q_T} \left(\int_0^T |v(\tau,x)| d\tau \right)^{p\lambda} dt dx,$$

where $0 < \lambda = \frac{p-r_0-1}{p-r-1} < 1$. By using Hölder's inequality with exponents $p_1 = \frac{1}{\lambda} (> 1)$ and $q_1 = \frac{p_1}{p_1 - 1}$ we obtain:

$$\int_{Q_T} \left(\int_0^T |v(\tau,x)| d\tau \right)^{p\lambda} dt dx \le$$

$$\le const \cdot \left(\int_{Q_T} \left(\int_0^T |v(\tau,x)| d\tau \right)^{p\lambda \frac{1}{\lambda}} dt dx \right)^{\lambda} \cdot \left(\int_{Q_T} 1^{q_1} \right)^{\frac{1}{q_1}} =$$

$$= const \cdot \left(\int_{Q_T} \left(\int_0^T |v(\tau,x)| d\tau \right)^{p} dt dx \right)^{\lambda}$$

Now we may estimate again by Hölder's inequality and after that we may use Fubini's theorem. We get

$$\int_{Q_T} \left(\int_0^T |v(\tau,x)| d\tau \right)^{p} dt dx \le$$

$$\le \int_{Q_T} \left[\left(\int_0^T |v(\tau,x)|^p d\tau \right)^{\frac{1}{p}} \left(\int_0^T 1^q d\tau \right)^{\frac{1}{q}} \right]^p dt dx =$$

$$= const \cdot \int_{Q_T} \int_0^T |v(\tau, x)|^p d\tau\, dx\, dt = const \cdot \int_{Q_T} |v(t,x)|^p dt\, dx \le const \cdot \|v\|_X^p.$$

Summarizing the above estimations one gets

$$\int_{Q_T} |[\tilde{G}_1(v)](t,x)|^{\frac{p}{p\;\; r\;\; 1}} dt\, dx \le const \cdot \|v\|_X^{p\lambda}.$$

From this it is easy to see that $\tilde{G}_1$ is a bounded operator which maps to $L^{\frac{p}{p\;\; r\;\; 1}}(Q_T)$. Further

$$\lim_{\|v\|_X \to \infty} \frac{\int_{Q_T} |[\tilde{G}_1(v)](t,x)|^{\frac{p}{p\;\; r\;\; 1}} dt\, dx}{\|v\|_X^p} = \lim_{\|v\|_X \to \infty} \|v\|_X^{p(\lambda\;\;\; 1)} = 0,$$

since $\lambda \;\;\; 1 < 0$. Continuity of the operator can be proved similarly to the previous theorem.

REMARK. From lemma it is easy too see that linear combinations of the above operators fulfil condtitions K5/a and K5/b, too.

3.2.3. Functions $b_i^{(l)}$, $d_i^{(l)}$

We show the well known examples. Let $b_i^{(l)}(t,x,\zeta_0,\zeta) := \tilde{b}_i^{(l)}(t,x,\zeta_0,\zeta_i^{(l)})$, where $\tilde{b}_i^{(l)}: Q_T \times \mathbb{R}^{N+1} \to \mathbb{R}$ is a Carathéodory function such that the following hold. Function $\zeta_i^{(l)} \mapsto \tilde{b}_i^{(l)}(t,x,\zeta_0,\zeta_i^{(l)})$ is strictly increasing,

$$\left|\tilde{b}_i^{(l)}\left(t,x,\zeta_0,\zeta_i^{(l)}\right)\right| \le c_1 \left(|\zeta_0|^{p\;\;\; 1} + \left|\zeta_i^{(l)}\right|^{p\;\;\; 1}\right) + k_1(t,x),$$

and

$$\tilde{b}_i^{(l)}\left(t,x,\zeta_0,\zeta_i^{(l)}\right)\zeta_i^{(l)} \ge c_2 \left|\zeta_i^{(l)}\right|^p \quad k_2(t,x),$$

where $c_1 > 0$, $k_1 \in L^q(\Omega)$ and $k_2 \in L^1(Q_T)$. Then $b_i^{(l)}$ obviously fulfil K1, K2/a. K4/a follows by inequality $|a+b|^s \le 2^{s\;\;\; 1}(|a|^s + |b|^s)$ and K3/a follows from monotonicity.

Similarly, let $d_i^{(l)}(t,x,\zeta_0,\zeta) := \tilde{d}_i^{(l)}(t,x,\zeta_0,\zeta_i^{(l)})$ $(i \ne 0)$ where $\tilde{d}_i^{(l)}: Q_T \times \mathbb{R}^{N+1} \to \mathbb{R}$ is a Carathéodory function such that the following hold. Function $\zeta_i^{(l)} \mapsto \tilde{d}_i^{(l)}(t,x,\zeta_0,\zeta_i^{(l)})$ is monotone nondecreasing, $\tilde{d}_i^{(l)}(t,x,\zeta_0,0) = 0$, and

$$\left|\tilde{d}_i^{(l)}\left(t,x,\zeta_0,\zeta_i^{(l)}\right)\right| \le c_1 \left(|\zeta_0|^r + \left|\zeta_i^{(l)}\right|^r\right) + k_1(t,x),$$

where $k_1 \in L^q(Q_T)$ and $0 \le r < p \quad 1$. If $i = 0$, then let $d_0^{(l)}$ be a Carathéodory-function which satisfies

$$\left|d_0^{(l)}(t,x,\zeta_0,\zeta)\right| \le c_1 \quad |\zeta_0|^r + |\zeta|^r) + k_1(t,x).$$

Then conditions K1, K2/b, K3/b obviously hold. To prove K4/b we only have to observe that (if $i \neq 0$) $\tilde{d}_i^{(l)}(t,x,\zeta_0,\zeta_i^{(l)})\zeta_i^{(l)} \ge 0$.

REMARK. The simplest examples for the above general conditions are $\zeta_i^{(l)} \mapsto \zeta_i^{(l)}|\zeta_i^{(l)}|^{p \quad 2}$ and $\zeta_i^{(l)} \mapsto \zeta_i^{(l)}|\zeta_i^{(l)}|^{r \quad 1}$ if $r > 0$. If $r = 0$ let $d_i^{(l)} \equiv 0$ and $d_0^{(l)} \equiv 1$.

References

[1] L. SIMON: On parabolic functional differential equations of general divergence form, *submitted to Proceedings of FSDONA 04*.

[2] L. SIMON: On systems of strongly nonlinear parabolic functional differential equations, *Periodica Math. Hung.* **33** (1996), 145–151.

[3] J. BERKOVITS and V. MUSTONEN: Topological degree for perturbations of linear maximal monotone mappings and applications to a class of parabolic problems, *Rend. Mat. Ser.* **VII,12** (1992), Roma, 597–621.

[4] J. L. LIONS: *Quelques méthodes de résolution des problémes aux limites non linéaires*, Dunod, Gauthier-Villars, Paris, 1969.

[5] E. ZEIDLER: *Nonlinear functional analysis and its applications II A and II B*, Springer, 1990.

Ádám Besenyei
Department: ....
Institute: ....
Address: ....
badam@cs.elte.hu