On a system consisting of three different types of differential equations

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Plan of the talk

- Motivation: *fluid flow in porous medium, nonlocal problems*
- The problem: *assumptions, weak formulation*
- Results: *existence of weak solutions in* $(0, T)$ *and in* $(0, \infty)$, *long-time behaviour, examples*
- Tool: *Schauder fixed point theorem, operators of monotone type*

Motivation - fluid flow

Flow in porous medium with reaction:

- solid matrix with holes (limestone)
- chemical reaction $\Longrightarrow$ porosity changes

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$$\omega D_t u = D_x\left[\alpha v D_x u\right] - v D_x u - uF(\omega)$$

$$D_t \omega = uF(\omega)$$

$$D_x\left[K(w) D_x p\right] = uF(\omega)$$

$$v = -K(\omega) D_x p, \qquad t > 0, x \in (0,1),$$

+ initial & boundary values, where ω porosity, u concentration, v velocity, p pressure [J. Logan et al. 2002]

Motivation - nonlocal problems

Nonlocal models:

- climatology, population dynamics
- nonlocal diffusion:

$$D_t u - \mathrm{div}(g(\int_\Omega u)Du) = f$$

  [M. Chipot, L. Molinet, 2001]
- results on quasilinear equations, systems [L. Simon]

The problem

A system consisting of three different types of DEs:

$$D_t u(t,x) - \sum_{i=1}^{n} D_i \left[a_i(t,x,\omega,u,Du,\mathbf{p},D\mathbf{p};\omega,u,\mathbf{p})\right] + a_0(t,x,\omega,u,Du,\mathbf{p},D\mathbf{p};\omega,u,\mathbf{p}) = g(t,x),\ \ u(0,x) = 0, \tag{1}$$

$$D_t \omega(t,x) = f(t,x,\omega,u;u), \quad \omega(0,x) = \omega_0(x), \tag{2}$$

$$-\sum_{i=1}^{n} D_i[b_i(t,x,\omega,u,\mathbf{p},D\mathbf{p};\omega,u)] + b_0(t,x,\omega,u,\mathbf{p},D\mathbf{p};\omega,u) = h(t,x). \tag{3}$$

Operators of monotone type

- X is a Banach space, X^* its dual, $\langle \cdot\, , \cdot \rangle$ the pairing between X^* and X

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- X is a Banach space, X^* its dual, $\langle \cdot\, , \cdot \rangle$ the pairing between X^* and X
- $A\colon X \to X^*$ is a *monotone* operator if $\langle A(u) - A(v), u - v \rangle \geq 0$ for every $u, v \in X$
- weaker condition $\rightsquigarrow$ *pseudomonotone* operators
- stronger condition $\rightsquigarrow$ *strictly, uniformly* monotone operators (uniqueness)

Notation

- $\Omega \subset \mathbb{R}^n,\ 0 < T < \infty,\ Q_T = (0,T) \times \Omega$
- $V_i = W^{1,p_i}(\Omega), X_i = L^{p_i}(0,T;V_i)\ (i = 1,2)$ with

$$\|u\|_{L^{p_i}(0,T;W^{1,p_i}(\Omega))} = \left(\int_0^T \|u(t)\|_{W^{1,p_i}(\Omega)}^{p_i} dt \right)^{1/p_i}$$

  where $2 \le p_1, p_2 < \infty$
- $\xi, \zeta_0, \zeta, \eta_0, \eta$ refer to $\omega, u, Du, \mathbf{p}, D\mathbf{p}$

Assumptions - function f

- Carathéodory

  $f(t, x, \xi, \zeta_0; v_1)$ is measurable in (t, x) and continuous in the other variables for fixed $v_1 \in L^{p_1}(Q_T)$

Assumptions - function f

- Carathéodory
- Lipschitz condition

  there exist a bounded $\mathcal{K}\colon L^{p_1}(Q_T) \to \mathbb{R}^+$, a continuous $K\colon \mathbb{R} \to \mathbb{R}^+$ satisfying $|K_1(\zeta_0)| \le d(|\zeta_0|^{p_1} + 1)$ such that

$$|f(t,x,\xi,\zeta_0;v_1) - f(t,x,\tilde{\xi},\zeta_0;v_1)| \le \mathcal{K}(v_1)K(\zeta_0) \cdot |\xi - \tilde{\xi}|.$$

Assumptions - function f

- Carathéodory
- Lipschitz condition
- “sign condition”

  there exists $\omega^* \in L^\infty(\Omega)$ such that

$$(\xi - \omega^*(x)) \cdot f(t, x, \xi, \zeta_0; v_1) \le 0$$

Assumptions - function f

- Carathéodory
- Lipschitz condition
- “sign condition”
- “continuity” in the nonlocal variable

$$u_k \longrightarrow u \text{ strongly in } L^{p_1}(Q_T)$$

$$\Downarrow$$

$$\lim_{k\to\infty} \|f(\cdot, \omega_k, u_k; u_k) - f(\cdot, \omega_k, u_k; u)\|_{L^1(Q_T)} = 0$$

Assumptions - function f

- Carathéodory
- Lipschitz condition
- “sign condition”
- “continuity” in the nonlocal variable

Assumptions - functions a_i, b_i

- Carathéodory

  $a_i(t, x, \xi, \zeta_0, \zeta, \eta_0, \eta; w, v_1, v_2)$ is measurable in (t, x) and continuous in the other variables for fixed $w \in L^2(Q_T), v_1 \in L^{p_1}(Q_T), v_2 \in X_2$:

Assumptions - functions a_i, b_i

- Carathéodory
- growth

there exist bounded $\mathfrak{c}_1 : L^2(Q_T) \times L^{p_1}(Q_T) \times X_2 \longrightarrow \mathbb{R}^+$, $k_1 : L^2(Q_T) \times L^{p_1}(Q_T) \times X_2 \longrightarrow L^{q_1}(Q_T)$ such that

$$|a_i(t, x, \xi, \zeta_0, \zeta, \eta_0, \eta; w, v_1, v_2)| \leq \mathfrak{c}_1(w, v_1, v_2) \left(|\zeta_0|^{p_1 - 1} + |\zeta|^{p_1 - 1} + |\eta_0|^{\frac{p_2}{q_1}} + |\eta|^{\frac{p_2}{q_1}} + [k_1(w, v_1, v_2)](t, x) \right)$$

Assumptions - functions a_i, b_i

- Carathéodory
- growth
- monotonicity

$$\sum_{i=1}^{n} \Big(a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2) - a_i(t,x,\xi,\zeta_0,\tilde{\zeta},\eta_0,\eta;w,v_1,v_2) \Big) (\zeta_i - \tilde{\zeta}_i)$$

$$\geq C \cdot \Big(|\zeta_0 - \tilde{\zeta}_0|^{p_1} + |\zeta - \tilde{\zeta}|^{p_1} \Big)$$

Assumptions - functions a_i, b_i

- Carathéodory
- growth
- monotonicity
- coercivity

there exist $c_2 > 0$, bounded $k_2 \colon L^{p_1}(Q_T) \to L^1(Q_T)$ such that

$$\sum_{i=0}^{n} a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2)\zeta_i \geq c_2\left(|\zeta_0|^{p_1} + |\zeta|^{p_1}\right) - [k_2(v_1)](t,x)$$

where $$\lim_{r\to\infty} \sup_{\|v_1\|_{L^{p_1}(Q_T)}\leq r} \frac{\|k_2(v_1)\|_{L^1(Q_T)}}{r^{p_1}} = 0.$$

Assumptions - functions a_i, b_i

- Carathéodory
- growth
- monotonicity
- coercivity
- "continuity" in the nonlocal part

$$\underbrace{\begin{array}{l} \omega_k \to \omega \text{ strongly in } L^2(Q_T) \\ u_k \to u \text{ strongly in } L^{p_1}(Q_T) \\ \mathbf{p}_k \to \mathbf{p} \text{ strongly in } X_2 \end{array}}$$

$$\Downarrow$$

$$a_i(t, x, \omega_k, \zeta_0, \zeta, \mathbf{p}_k, D\mathbf{p}_k; \omega_k, u_k, \mathbf{p}_k) \to a_i(t, x, \omega, u, Du, \mathbf{p}, D\mathbf{p}; \omega, u, \mathbf{p})$$

Assumptions - functions a_i, b_i

- Carathéodory
- growth
- monotonicity
- coercivity
- “continuity” in the nonlocal part

- similarly for b_i: $p_1 \rightsquigarrow p_2$...

The problem - weak formulation

- Define $A\colon L^\infty(Q_T)\times X_1\times X_2\to X_1^*$

$$[A(\omega,u,\mathbf{p}),v_1]:=$$

$$\int_{Q_T}\sum_{i=1}^{n} a_i(t,x,\omega(t,x),u(t,x),Du(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p})D_i v_1(t,x)\,dt\,dx$$

$$+\int_{Q_T} a_0(t,x,\omega(t,x),u(t,x),Du(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p})v_1(t,x)\,dt\,dx$$

- similarly $B\colon L^\infty(Q_T)\times X_1\times X_2\to X_2^*$

The problem - weak formulation

- Weak form:

$$\begin{aligned} Lu + A(\omega, u, \mathbf{p}) &= G \\ \omega(t,x) &= \omega_0(x) + \int_0^t f(s, x, \omega(s,x), u(s,x); u)\, ds \\ B(\omega, u, \mathbf{p}) &= H. \end{aligned}$$

where $(t,x) \in Q_T$, $G \in X_1^*$, $H \in X_2^*$, $L = D_t$

$$D(L) = \{u \in X_1 : D_t u \in X_1^*, u(0) = 0\}.$$

Results - existence in $(0, T)$

<u>THEOREM</u>: previous assumptions
$\Rightarrow$ there exist solutions $\omega \in L^\infty(Q_T)$,
$u \in L^{p_1}(0, T; W^{1,p_1}(\Omega)), \mathbf{p} \in L^{p_2}(0, T; W^{1,p_2}(\Omega))$

Results - existence in $(0, T)$

Proof: Schauder fixed point theorem

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Define $\tilde{A} \colon L^2(Q_T) \times L^{p_1}(Q_T) \times X_2 \to X_1^*$

$$[\tilde{A}(\tilde{u}, \omega, u, \mathbf{p}), v_1] :=$$

$$\int_{Q_T} \sum_{i=1}^{n} a_i(t,x,\omega(t,x),\tilde{u}(t,x),D\tilde{u}(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p}) D_i v_1(t,x)\,dt\,dx$$

$$+ \int_{Q_T} a_0(t,x,\omega(t,x),\tilde{u}(t,x),D\tilde{u}(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p}) v_1(t,x)\,dt\,dx$$

Results - existence in $(0, T)$

Proof: Schauder fixed point theorem

Consider the modified problem

$$L\tilde{u} + \tilde{A}(\tilde{u}, \omega, u, \mathbf{p}) = G \tag{4}$$

$$\tilde{\omega}(t, x) = \omega_0(x) + \int_0^t f(s, x, \tilde{\omega}(s, x), \tilde{u}(s, x); \tilde{u})\, ds \tag{5}$$

$$B(\tilde{\omega}, \tilde{u}, \tilde{\mathbf{p}}) = H. \tag{6}$$

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1. for fixed $\omega \in L^2(Q_T), u \in L^{p_1}(Q_T), \mathbf{p} \in X_2$ there exists unique solution $\tilde{u} \in X_1$ of (1)

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1. for fixed $\omega \in L^2(Q_T), u \in L^{p_1}(Q_T), \mathbf{p} \in X_2$ there exists unique solution $\tilde{u} \in X_1$ of (1)
2. with this $\tilde{u}$ we solve (2), we obtain $\tilde{\omega} \in L^\infty(Q_T)$

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1. for fixed $\omega \in L^2(Q_T), u \in L^{p_1}(Q_T), \mathbf{p} \in X_2$ there exists unique solution $\tilde{u} \in X_1$ of (1)
2. with this $\tilde{u}$ we solve (2), we obtain $\tilde{\omega} \in L^\infty(Q_T)$
3. finally, from (3) we obtain unique $\tilde{\mathbf{p}} \in X_2$

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So we may define $(\omega, u, \mathbf{p}) \xrightarrow{\Phi} (\tilde{\omega}, \tilde{u}, \tilde{\mathbf{p}})$

$\Phi : L^2(Q_T) \times L^{p_1}(Q_T) \times X_2 \longrightarrow L^2(Q_T) \times L^{p_1}(Q_T) \times X_2$

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So we may define $(\omega, u, \mathbf{p}) \xrightarrow{\Phi} (\tilde{\omega}, \tilde{u}, \tilde{\mathbf{p}})$
$\Phi : L^2(Q_T) \times L^{p_1}(Q_T) \times X_2 \longrightarrow L^2(Q_T) \times L^{p_1}(Q_T) \times X_2$

One may show Φ is continuous, compact, there is a ball $B(0, r)$ which is mapped into itself by Φ.

Results - existence in $(0, T)$

Proof: Schauder fixed point theorem

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So we may define $(\omega, u, \mathbf{p}) \xrightarrow{\Phi} (\tilde{\omega}, \tilde{u}, \tilde{\mathbf{p}})$

$\Phi\colon\, L^2(Q_T) \times L^{p_1}(Q_T) \times X_2 \longrightarrow L^2(Q_T) \times L^{p_1}(Q_T) \times X_2$

A fixed point of Φ is a desired solution. $\square$

Results - existence in $(0, \infty)$

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THEOREM: f, a_i, b_i are of Volterra type in Q_∞

the nonlocal part does not depend on the future

Results - existence in $(0, \infty)$

By using a "diagonal process" one may extend the previous theorem to $(0, \infty)$

<u>THEOREM</u>: f, a_i, b_i are of Volterra type in Q_∞ $\Longrightarrow$ there exist solutions $\omega \in L^\infty(Q_\infty), u \in L^{p_1}_{\mathrm{loc}}(0, \infty; W^{1,p_1}(\Omega)), \mathbf{p} \in L^{p_2}_{\mathrm{loc}}(0, \infty; W^{1,p_2}(\Omega))$

Results - long-time behaviour

- Boundedness

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  bounded right hand side
  $G \in L^\infty(0,\infty;V_1^*), H \in L^\infty(0,\infty;V_2^*)$

  $$\Downarrow$$

  $\omega \in L^\infty(Q_\infty), u \in L^\infty(0,\infty;L^2(\Omega)), \mathbf{p} \in L^\infty(0,\infty;W^{1,p_2}(\Omega))$

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- Stabilization as $t \longrightarrow \infty$

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- Boundedness
- Stabilization as $t \to \infty$

(i) previous assumptions $p_1 = p_2 = p$

(ii) $\omega^* \in L^\infty(\Omega)$ exponentially stable

$$(\xi - \omega^*(x))f(t, x, \xi, \zeta_0; v_1) \leq -m(\xi - \omega^*(x))^2$$

Results - long-time behaviour

- Boundedness
- Stabilization as $t \to \infty$

(i) previous assumptions $p_1 = p_2 = p$

(ii) $\omega^* \in L^\infty(\Omega)$ exponentially stable

(iii) stabilization of a_i, b_i, G, H

$$\lim_{\substack{t\to\infty\\ \xi\to\xi^*}} a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2) = a_{i,\infty}(x,\xi^*,\zeta_0,\zeta,\eta_0,\eta)$$

$$\lim_{\substack{t\to\infty\\ \xi\to\xi^*}} b_i(t,x,\xi,\zeta_0,\eta_0,\eta;w,v_1) = b_{i,\infty}(x,\xi^*,\zeta_0,\eta_0,\eta)$$

$$\lim_{t\to\infty} \|F(t) - F_\infty\|_{W^{1,p}(\Omega)^*} = 0, \qquad \lim_{t\to\infty} \|G(t) - G_\infty\|_{W^{1,p}(\Omega)^*} = 0$$

Results - long-time behaviour

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- Stabilization as $t \to \infty$

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(iii) stabilization of a_i, b_i, G, H

$$\Downarrow$$

$$\|\omega(t,\cdot) - \omega^*\|_{L^\infty(\Omega)} \leq \|\omega_0\|_{L^\infty(\Omega)} e^{-mt}, \quad u(t) \stackrel{t\to\infty}{\longrightarrow} u_\infty \text{ in } L^2(\Omega),$$

$$\int_{t-1}^{t+1} \|u(s) - u_\infty\|^p_{W^{1,p}(\Omega)} ds \stackrel{t\to\infty}{\longrightarrow} 0, \int_{t-1}^{t+1} \|p(s) - p_\infty\|^p_{W^{1,p}(\Omega)} ds \stackrel{t\to\infty}{\longrightarrow} 0,$$

where $u_\infty, \mathbf{p}_\infty \in W^{1,p}(\Omega)$ are unique solutions of the stationary problem $A_\infty(\omega^*, u_\infty, p_\infty) = G_\infty, B_\infty(\omega^*, u_\infty, p_\infty) = H_\infty$.

Results - long-time behaviour

- Boundedness
- Stabilization as $t \longrightarrow \infty$
- Asymptotics

$$\Big\| a_i(t,\cdot,\omega(t,\cdot),u(\cdot),Du(\cdot),\mathbf{p}(\cdot),D\mathbf{p}(\cdot);w,v_1,v_2) - a_{i,\infty}(\cdot,\omega^*(\cdot),u(\cdot),Du(\cdot),\mathbf{p}(\cdot),D\mathbf{p}(\cdot))\Big\|^q_{L^q(\Omega)} \le k^* t^{-\beta},$$

$$\Big\| b_i(t,\cdot,\omega(t,\cdot),u(\cdot),\mathbf{p}(\cdot),D\mathbf{p}(\cdot);w,v_1) - b_{i,\infty}(\cdot,\omega^*(t,\cdot),u(\cdot),\mathbf{p}(\cdot),D\mathbf{p}(\cdot))\Big\|^q_{V^*} \le k^* t^{-\beta},$$

$$\|G(t) - G_\infty\|^q_{V^*} \le k^* t^{-\beta}, \quad \|H(t) - H_\infty\|^q_{V^*} \le k^* t^{-\beta}.$$

Results - long-time behaviour

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$$\Big\| a_i(t,\cdot,\omega(t,\cdot),u(\cdot),Du(\cdot),\mathbf{p}(\cdot),D\mathbf{p}(\cdot);w,v_1,v_2) - a_{i,\infty}(\cdot,\omega^*(\cdot),u(\cdot),Du(\cdot),\mathbf{p}(\cdot),D\mathbf{p}(\cdot))\Big\|_{L^q(\Omega)}^q \le k^* t^{-\beta},$$

$$\Big\| b_i(t,\cdot,\omega(t,\cdot),u(\cdot),\mathbf{p}(\cdot),D\mathbf{p}(\cdot);w,v_1) - b_{i,\infty}(\cdot,\omega^*(t,\cdot),u(\cdot),\mathbf{p}(\cdot),D\mathbf{p}(\cdot))\Big\|_{V^*}^q \le k^* t^{-\beta},$$

$$\|G(t)-G_\infty\|_{V^*}^q \le k^* t^{-\beta}, \quad \|H(t)-H_\infty\|_{V^*}^q \le k^* t^{-\beta}.$$

$$\Downarrow$$

$$\int_t^\infty \|u(s)-u_\infty\|_V^{2\alpha}\,\mathrm{d}s + \int_t^\infty \|\mathbf{p}(t)-\mathbf{p}_\infty\|_V^{2\alpha}\,\mathrm{d}s \le \mathrm{const}\cdot t^{\frac{1}{1-\alpha}}$$

holds for $t>0$ sufficiently large and for $\alpha = \max\left\{\frac{p}{2}, 1+\frac{1}{\beta-1}\right\}$.

Results - long-time behaviour

- Boundedness
- Stabilization as $t \longrightarrow \infty$
- Asymptotics

Idea of the proof: one may deduce

$$y'(t) + \mathrm{const} \cdot y(t)^{\frac{p}{2}} + \mathrm{const} \cdot \|\mathbf{p}(t) - \mathbf{p}_\infty\|_V^p \leq \varphi(t)$$

where $y(t) = \|u(t) - u_\infty\|^2_{L^2(\Omega)}$ and $\varphi(t) \longrightarrow 0$

Results - long-time behaviour

- Boundedness
- Stabilization as $t \to \infty$
- Asymptotics

Examples

$$D_t\omega + u(\omega - \omega^*) f_1(t,x)\varphi(u) = 0$$

$$D_t u - \operatorname{div}\left(\psi(\mathbf{p})|Du|^{p_1-2}Du\right) + \psi(\mathbf{p})u|u|^{p_1-2} = g(t,x)$$

$$-\operatorname{div}\left(\pi(\omega)|D\mathbf{p}|^{p_2-2}D\mathbf{p}\right) + \pi(\omega)\mathbf{p}|\mathbf{p}|^{p_2-2} = h(t,x),$$

where, e.g., to obtain existence

$$[\varphi(u)](t) = \Phi\left(\left[\int_{Q_t} |u(s,\xi)|^2 ds\, d\xi\right]^{\frac{1}{2}}\right),$$

$$[\psi(\mathbf{p})](t) = \Psi\left(\left[\int_0^t |D\mathbf{p}(s,x)|^{p_2} ds\right]^{\frac{1}{p_2}}\right),$$

$$[\pi(\omega)](t) = \Pi\left(\int_\Omega |\omega(t,\xi)|\, ds\, d\xi\right).$$

Examples

$$D_t\omega + u(\omega - \omega^*)f_1(t,x)\varphi(u) = 0$$

$$D_t u - \operatorname{div}\,(\psi(\mathbf{p})|Du|^{p_1-2}Du) + \psi(\mathbf{p})u|u|^{p_1-2} = g(t,x)$$

$$-\operatorname{div}\,(\pi(\omega)|D\mathbf{p}|^{p_2-2}D\mathbf{p}) + \pi(\omega)\mathbf{p}|\mathbf{p}|^{p_2-2} = h(t,x),$$

where, e.g., to obtain stabilization

$$[\pi(w)](t,x) = \chi(t)\int_\Omega |w(s,x)|^\beta + \pi^*(x),$$

with $\lim_{t\to\infty}\chi(t) = 0$, $\pi^* \in L^\infty(\Omega)$ and similarly for the other operators;

to obtain estimates $\chi(t) \le \mathrm{const}\cdot t^{-\beta}$.

The end

Thank you for your attention!