Bounds for the expected value of one-step processes

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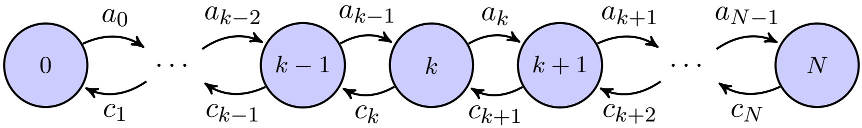
Outline

1. Formulation of the problem
    - One-step processes
    - Mean-field approximation and accuracy
2. Towards the main result
    - Moments of the process
    - Approximating system
    - Main result, tools and idea of the proof
3. Applications
    - SIS epidemic propagation
    - SIS epidemic with airborne infection
    - Voter model

Setting

- Continuous-time Markov process $X(t)$ with state space $\{0, 1, \ldots, N\}$.
- One-step or Birth–Death process: transition from state k is possible only to state $k-1$ at rate c_k and to state $k+1$ at rate a_k (and let $a_{-1} = a_N = c_0 = c_{N+1} = 0$).



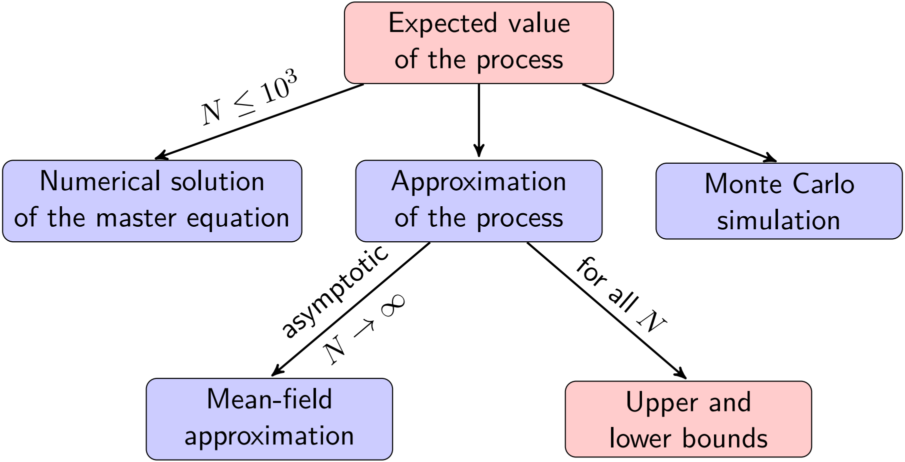
- Example: SIS epidemic or voter model.
- Probability distribution of $X(t)$: $p_k(t) = P[X(t) = k]$.

$$\Downarrow$$

- Time-evolution is described by Kolmogorov or master equation:

$$p_k' = a_{k-1}p_{k-1} - (a_k + c_k)p_k + c_{k+1}p_{k+1} \quad (k = 0, \ldots, N).$$

Expected value of the process



Accuracy of the mean-field approximation

- T. G. Kurtz (1970, 2005): density-dependent Markov processes, stochastic convergence of the exact stochastic to the deterministic mean-field model. (Tools: Trotter-type results, martingale theory.)
- A. Bátkai, I. Z. Kiss, E. Sikolya, P. L. Simon (2012): uniform convergence of the expected value to the mean-field model, precisely, if $y_1(t) = \mathbb{E}[X(t)/N]$ is the expected value of the scaled process and $y(t)$ is the mean-field approximation, then

$$|y_1(t) - y(t)| \leq \frac{C_T}{N} \quad (t \in [0, T]).$$

  (Tools: operator semigroups.)
- B. Armbruster, E. Beck (2015): SIS epidemic model on a complete graph, upper and lower bounds for y_1. (Tools: elementary ODE.)
- B. Armbruster, Á. B., P. L. Simon (2015): extension of the results of Armbruster & Beck to a wider class of Markov chains.

Formulation of the main result: ODEs for the moments

- Transition rates are density dependent polynomials:

$$\frac{a_k}{N} = \sum_{j=0}^{m} A_j (k/N)^j \quad \text{and} \quad \frac{c_k}{N} = \sum_{j=0}^{m} C_j (k/N)^j$$

such that $A(1) = 0$ and $C(0) = 0$ hold.

$$\Downarrow$$

- ODEs for the scaled moments: $y_n = \mathbb{E}[X(t)/N]$ $(n = 0, 1, 2, \dots)$.

$$y_1' = \sum_{j=0}^{m} D_j y_j,$$

$$y_n' = n \sum_{j=0}^{m} D_j y_{n+j-1} + \frac{1}{N} R_n \quad (n = 2, 3, \dots),$$

where $D_j = A_j - C_j$, $0 \leq R_n \leq \frac{n(n-1)}{2} c$, and $c = \sum_{j=0}^{m} (|A_j| + |C_j|)$.

Formulation of the main result: approximations

- Mean-field equation: we approximate $y_n \approx y_1^n$, then by $y_1' = \sum_{j=0}^{m} D_j y_j$,

$$y' = \sum_{j=0}^{m} D_j y^j, \quad y(0) = y_1(0).$$

- ODEs for the powers of y:

$$(y^n)' = n \sum_{j=0}^{m} D_j (y^n)^{\frac{n+j-1}{n}}, \quad y^n(0) = y_1^n(0) \quad (n = 2, 3, \dots).$$

- Approximating system for the moments:

$$z_1' = \sum_{j=0}^{m} D_j z_j, \quad z_1(0) = y_1(0),$$

$$z_n' = n \sum_{j=0}^{m} D_j z_n^{\frac{n+j-1}{n}} + \frac{n(n-1)}{2N} c, \; z_n(0) = y_1^n(0) \quad (n = 2, 3, \dots, m),$$

where we let $z_0 = 1$.

Main result

Theorem (B. Armbruster, Á. B., P. L. Simon, 2015)

Assume that

$$D_0 \geq 0, D_1 \in \mathbb{R} \text{ and } D_j \leq 0 \text{ for } j \geq 2,$$

and let $y_1(0) = u \in (0, 1]$ *be fixed. Then for the solutions* y *of the mean-field equation and* z_1 *of the approximating system, it holds that*

$$z_1(t) \leq y_1(t) \leq y(t) \text{ for } t \geq 0$$

and for every $T > 0$ *there exists a constant* $C_T > 0$ *such that*

$$|z_1(t) - y(t)| \leq \frac{C_T}{N} \text{ in } [0, T].$$

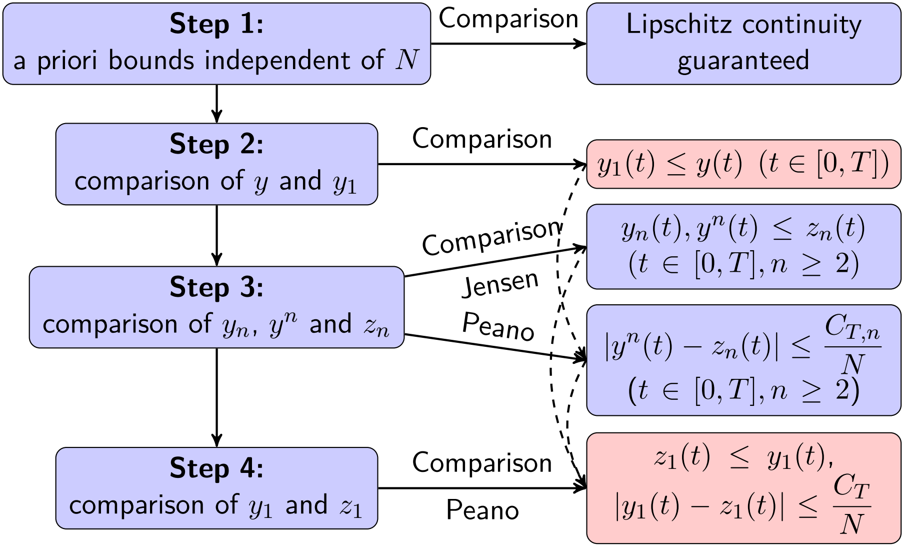
Idea of the proof.

Smart application of three familiar inequalities. □

Tools of the proof: familiar inequalities

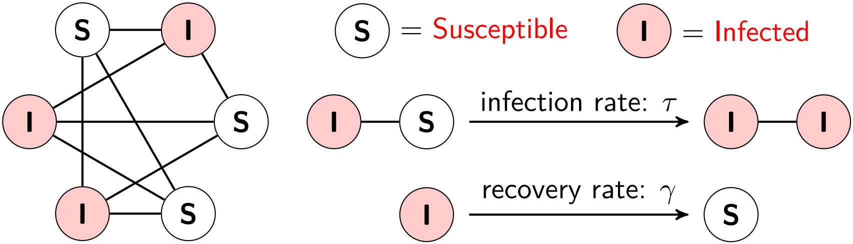
- Comparison: If $f\colon [0,T]\times[a,b]\to\mathbb{R}$ is Lipschitz continuous in its second variable, then
$$\left.\begin{aligned} x_1'(t) &= f(t,x_1(t)), x_1(0)=x_0,\\ x_2'(t) &\le f(t,x_2(t)), x_2(0)\le x_0 \end{aligned}\right\} \implies x_2(t)\le x_1(t) \text{ for } t\in[0,T].$$
- Peano's inequality: Assume that $f_1, f_2\colon [0,T]\times[a,b]\to\mathbb{R}$ are Lipschitz continuous in their second variable with Lipschitz constant L and $|f_1(t,x)-f_2(t,x)|\le M$ in $[0,T]\times[a,b]$ with some M. Then
$$\left.\begin{aligned} x_1'(t) &= f_1(t,x_1(t)), x_1(0)=x_0,\\ x_2'(t) &= f_2(t,x_2(t)), x_2(0)= x_0 \end{aligned}\right\}$$
$$\implies |x_1(t)-x_2(t)|\le \frac{M}{L}\left(e^{Lt}-1\right) \text{ for } t\in[0,T].$$
- Jensen's inequality: If X is a random variable and $\varphi\colon\mathbb{R}\to\mathbb{R}$ is a convex function, then
$$\varphi(\mathbb{E}[X])\le\mathbb{E}[\varphi(X)].$$

Flow of the proof



Application: SIS epidemic propagation

- Random d-regular graph with N nodes in two possible states:



- State space $\approx \{0, 1, \ldots, N\}$ the number of infected nodes.
- Average number of SI edges is $(N - k) \cdot d\frac{k}{N}$.

$$\Downarrow$$

- Transition rates are $a_k = \tau d(N - k)\dfrac{k}{N}$ and $c_k = \gamma k$.

Application: SIS epidemic propagation

- Mean-field equation:

$$y' = (\tau d - \gamma)y - \tau d y^2, \quad y(0) = i/N.$$

- Approximating system:

$$z_1' = (\tau d - \gamma)z_1 - \tau d z_2, \quad z_1(0) = i/N,$$

$$z_2' = 2(\tau d - \gamma)z_2 - 2\tau d z_2^{3/2} + \frac{2\tau d + \gamma}{N}, \quad z_2(0) = (i/N)^2.$$

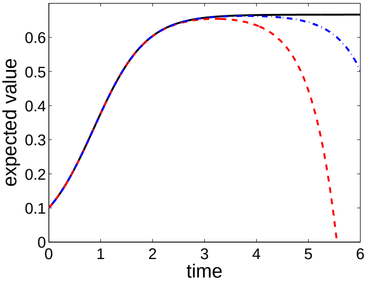


Figure: SIS epidemic propagation.
Parameters: $\gamma = 1$, $\tau = 0.1$, $d = 30$.
Curves: ——— y; - - - - z_1 for $N = 10^6$; -·-·-· z_1 for $N = 10^7$.

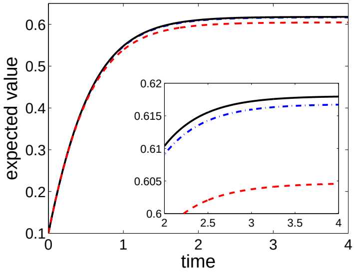
Application: SIS epidemic with airborne infection

- Infection also due to external forcing: $a_k = \tau d(N-k)\frac{k}{N} + \beta(N-k)$.
- Mean-field equation:

$$y' = \beta + (\tau d - \beta - \gamma)y - \tau d y^2, \quad y(0) = i/N.$$

- Approximating system:

$$z_1' = \beta + (\tau d - \beta - \gamma)z_1 - \tau d z_2, \quad z_1(0) = i/N,$$
$$z_2' = 2\beta z_2^{1/2} + 2(\tau d - \beta - \gamma)z_2 - 2\tau d z_2^{3/2} + \frac{c}{N}, \quad z_2(0) = (i/N)^2.$$

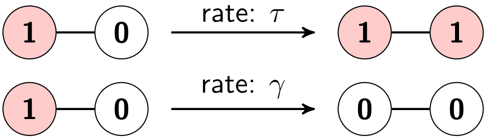


$$c = \beta + |\tau d - \beta| + \tau d + \gamma$$

Figure: SIS with airborne infection.
Parameters:
$N = 100, \gamma = 1, \tau = 0.05, d = 20, \beta = 1.$
Curves: ——— y; - - - - z_1; -·-·-· y_1.
Inset: stationary part of the curves.

Application: voter model

- Two opinions, **0** and **1** are changing:



$$\Downarrow$$

- Transition rates are $a_k = \tau d(N-k)\dfrac{k}{N}$ and $c_k = \gamma d k \dfrac{N-k}{N}$.
- Mean-field equation:

$$y' = (\tau d - \gamma d)(y - y^2), \quad y(0) = i/N.$$

- Approximating system:

$$z_1' = (\tau d - \gamma d) z_1 - (\tau d - \gamma d) z_2, \quad z_1(0) = i/N,$$

$$z_2' = 2(\tau d - \gamma d) z_2 - 2(\tau d - \gamma d) z_2^{3/2} + \frac{2\tau d + 2\gamma d}{N}, \quad z_2(0) = (i/N)^2.$$

Application: voter model

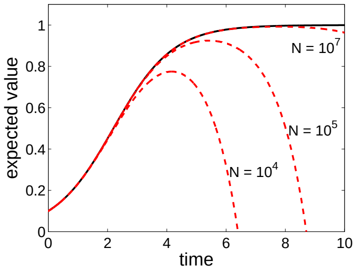


Figure: Voter-like model: case $D_2 < 0$.
Parameters: $\gamma = 0.1$, $\tau = 0.2$, $d = 10$.
Curves: —— y; - - - - z_1.

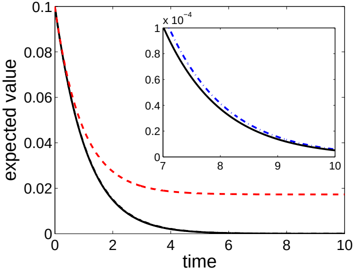


Figure: Voter-like model: case $D_2 > 0$.
Parameters:
$N = 200$, $\gamma = 0.2$, $\tau = 0.1$, $d = 10$.
Curves: —— y; - - - - z_1; -·-·- y_1.
Inset: stationary part of the curves.

Further directions

- Relaxing the sign condition: convexity arguments?
- Improving the lower bound with modified approximating systems:

1. $z_1' = D_0 + D_1 z_1 + D_2 z_2.$ ← $z_2^{q/2} z_1^{2-q}$

2. $z_2' = 2D_0 z_2^{1/2} + 2D_1 z_2 + 2D_2 z_2^{3/2} + \dfrac{2\tau d + \gamma}{N}.$ ← z_2^2 / z_1

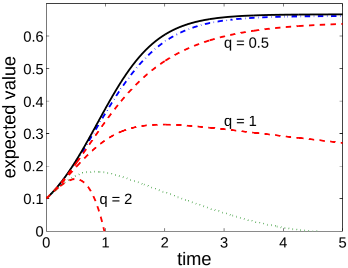


Figure: Lower bounds of the SIS model.
Parameters:
$N = 100,\ \gamma = 1,\ \tau = 0.1,\ d = 30.$
Curves: ——— y; -·-·- y_1;
- - - - z_1 in case 1; ······· z_1 in case 2.

References

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Thank you for your attention!