IMAGE PROBLEM CORNER: NEW PROBLEMS

Problems: We introduce 6 new problems in this issue and invite readers to submit solutions for publication in *IMAGE*. Solutions: We present solutions to all problems in the previous issue [*IMAGE* 49 (Fall 2012), p. 52] except problem 49.3, for which we still seek solutions. Submissions: Please submit proposed problems and solutions in macro-free LaTeX along with the PDF file by e-mail to *IMAGE* Problem Corner editor Bojan Kuzma (bojan.kuzma@famnit.upr.si). The working team of the Problem Corner consists of Gregor Dolinar, Shaun Fallat, Alexander Guterman, Nung-Sing Sze, and Rajesh Pereira.

New Problems:

Problem 50-1: An Adjugate Identity

Proposed by Khaled Aljanaideh, *University of Michigan, Ann Arbor, MI, USA*, khaledfj@umich.edu
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Let $\operatorname{adj} A \in M_n(\mathbb{C})$ denote the adjugate (transposed matrix of cofactors) of $A \in M_n(\mathbb{C})$, let $A_{[i,j]} \in M_{n-1}(\mathbb{C})$ denote A with the i-th row and j-th column removed, and let $A_{[i,\cdot]} \in M_{(n-1)\times n}(\mathbb{C})$ denote A with the i-th row removed. If $e_i \in \mathbb{C}^n$ is the i-th column vector of the standard basis, show that

$$\left[(\operatorname{adj} A)_{[i,\cdot]} + (\operatorname{adj} A_{[i,i]})A_{[i,\cdot]}\right] e_i = 0_{(n-1)\times 1}; \qquad i \in \{1,\dots,n\}.$$

Problem 50-2: Range-Hermitianness of Certain Functions of Projectors

Proposed by Oskar Maria Baksalary, *Adam Mickiewicz University, Poznań, Poland*, obaksalary@gmail.com
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(i) Let $P, Q \in M_n(\mathbb{C})$ be Hermitian idempotent matrices of order n. Show that $P + Q - PQ$ is range-Hermitian.

(ii) Let $R \in M_n(\mathbb{C})$ be an idempotent matrix of order n and let $R^\dagger$ be its Moore-Penrose inverse. Show that $I_n - R^\dagger$ is range-Hermitian, where I_n is the identity matrix of order n.

Recall that a matrix is range-Hermitian (also called EP) if its range, and the range of its conjugate transpose, coincide.

Problem 50-3: Trace Inequality for Positive Block Matrices

Proposed by Ádám Besenyei, *Department of Applied Analysis, Eötvös Loránd University, Hungary*, badam@cs.elte.hu

Let $A, B, C \in M_n(\mathbb{C})$ be such that the block matrix $\left(\begin{smallmatrix} A & B \\ B^* & C\end{smallmatrix}\right) \in M_{2n}(\mathbb{C})$ is positive-semidefinite. Show that

$$\operatorname{Tr}(AC) - \operatorname{Tr}(B^*B) \le \operatorname{Tr}(A)\operatorname{Tr}(C) - \operatorname{Tr}(B^*)\operatorname{Tr}(B).$$

Problem 50-4: Matrix Power Coefficients

Proposed by Moubinool Omarjee, *Lycée Henri IV, Paris, France*, ommou@yahoo.com

Find all real matrices A such that $A^r = (a_{ij}^r)$ for any integer r (A^r is the usual product of A r-times). What is the answer for matrices over commutative fields?

Problem 50-5: A Matrix of Divided Differences

Proposed by Rajesh Pereira, *University of Guelph, Guelph, Canada*, pereirar@uoguelph.ca

Let $p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n$ be an n-th degree monic polynomial with complex coefficients and let $\{x_k\}_{k=1}^n$ be a set of n pairwise distinct real numbers. Let M be the n–by–n matrix of first divided differences of p (so for $1 \le i, j \le n$ we have $m_{i,j} = (p(x_i) - p(x_j))/(x_i - x_j)$ and $m_{ii} = p'(x_i)$). Show that the determinant of M is positive if n is congruent to 0 or 1 mod 4 and negative if n is congruent to 2 or 3 mod 4.

Problem 50-6: Diagonalizable Matrices Over $\mathbb{F}_p$

Proposed by Denis Serre, *École Normale Supérieure de Lyon, France*, denis.serre@ens-lyon.fr

Let $p \ge 2$ be a prime number and let $\mathbb{F}_p$ denote the field $\mathbb{Z}/p\mathbb{Z}$. Prove that $A \in M_n(\mathbb{F}_p)$ is diagonalizable *within* $M_n(\mathbb{F}_p)$ if and only if $A^p = A$.

Solutions to Problems 49-1 *through* 49-6*, except* 49-3*, are on pp.* 36–43.