

Traveling waves and Taylor series: do they have something in common?

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It is always fascinating when two seemingly distant concepts of mathematics unexpectedly come across and it turns out that they are not as far from each other as it appears. Such two completely different notions might be at first sight traveling waves and Taylor series. In this note, we shall show that they do have something in common: a simple property of traveling waves can be used to derive the following well-known theorem from the theory of Taylor series.

Theorem 1. *Let $f : (a, b) \rightarrow \mathbb{R}$ be an infinitely differentiable function and suppose that there is $M > 0$ such that*

$$|f^{(n)}(x)| \leq M \text{ for } n = 0, 1, \dots \text{ and } x \in (a, b). \quad (1)$$

Then for $x, x + h \in (a, b)$,

$$f(x + h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \dots$$

so f is represented by its Taylor expansion around $x \in (a, b)$.

Waves on a rope

Let us first get acquainted with the notion of a traveling wave. It is a phenomenon that everyone, even non-scientists are familiar with. Assume that we take a stretched rope and we pluck it somewhere. Then a bump starts traveling along the rope. We can also observe that, by neglecting friction, the bump keeps the same shape and remains in the same plane during its move. This is called a traveling wave. Of course, in real life the bump slowly dissipates and as it reaches the end of the rope it is „reflected” but now we disregard these cases.

In order to describe traveling waves mathematically, let us choose the rope as the x -axis and fix the y -axis in the plane of the bump. If the shape of the bump is given by the function $u_0(x)$ at the initial time and it travels with constant velocity v (being negative if the bump travels to the left), then at later time t the shape of the rope is given by $u_0(x - vt)$, see Figure 1. Therefore, a traveling wave might be considered as a two-variable function $u(x, t) = u_0(x - vt)$ which gives the vertical displacement at time t of the point on the rope whose coordinate is x .

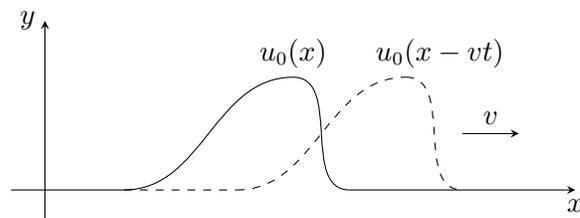


Figure 1. A traveling wave

Now, a simple question might arise: is there an equation that describes traveling waves? We can easily guess such an equation. Since $u(x, t) = u_0(x - vt)$, it follows that u is constant along the lines $x - vt = c$ in the (x, t) plane, see Figure 2. So the directional derivative of u must be zero along these lines which have direction vector $\vec{w} := (v, 1)$. Therefore, if u is continuously differentiable, then

$$\partial_{\vec{w}} u(x, t) = v\partial_x u(x, t) + \partial_t u(x, t) = 0.$$

The right-hand side of the above equation is traditionally written in the form

$$\partial_t u(x, t) + v\partial_x u(x, t) = 0$$

which is called the transport equation (see [2, Sec. 2.1]). The preceding arguments are reversible, if u is a continuously differentiable solution of the transport equation, then u is constant along the lines $x - vt = c$. So u must be of the form $u(x, t) = u_0(x - vt)$ with some continuously differentiable function u_0 .

More precisely, suppose that a continuously differentiable function $U(x, t)$ is a solution of the transport equation for $x \in (a, b)$ and $t \in (-T, T)$ where $T \in \mathbb{R}$. Let $c \in (a, b)$ be fixed, then by using the chain rule for multivariable functions we find that

$$\frac{d}{dt} U(c + vt, t) = v\partial_x U(c + vt, t) + \partial_t U(c + vt, t) = 0.$$

Consequently, the function $t \mapsto U(c + vt, t)$ is constant for $t \in (-T, T)$ such that $c + vt \in (a, b)$. Thus, $U(c + vt, t) = U(c, 0)$ and therefore $U(x, t) = U(x - vt, 0)$ for pairs (x, t) such that $x - vt \in (a, b)$.

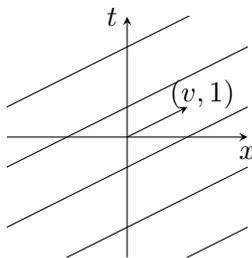


Figure 2. The lines $x - vt = c$

The rabbit out of the hat

Now we reveal how the preceding property of traveling waves appear in connection with Taylor series and prove the Theorem.

Let $f: (a, b) \rightarrow \mathbb{R}$ be an infinitely differentiable function such that (1) holds and define for $x \in (a, b)$ and $|t| \leq T$ the series

$$\psi(x, t) = f(x) + f'(x)t + \frac{f''(x)}{2!}t^2 + \frac{f'''(x)}{3!}t^3 + \dots$$

Then by termwise differentiation we obtain that

$$\partial_x \psi(x, t) = \partial_t \psi(x, t) = f'(x) + f''(x)t + \frac{f'''(x)}{2!}t^2 + \frac{f^{(4)}(x)}{3!}t^3 + \dots$$

Thus, ψ is the solution of the transport equation with $v = -1$. Since $\psi(x, 0) = f(x)$, it follows that $\psi(x, t) = \psi(x + t, 0) = f(x + t)$ for $x + t \in (a, b)$. Replacing t with h , we obtain the Taylor series representation of f around $x \in (a, b)$:

$$f(x + h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \dots$$

To be fully precise, we have to guarantee that ψ is termwise differentiable in both variables. This follows by the assumption (1). Then

$$\left| f(x) + f'(x)t + \frac{f''(x)}{2!}t^2 + \dots \right| \leq M \left(1 + T + \frac{T^2}{2!} + \frac{T^3}{3!} + \dots \right) = Me^T$$

thus the series of ψ is absolutely and uniformly convergent due to the Weierstrass M-test. Analogously, the formal partial derivatives of ψ are also absolutely and uniformly convergent and these properties finally ensure the termwise differentiability of ψ (see [3, Thm. 7.17]). So the proof of the Theorem is complete.

The above proof is certainly not new, the idea dates back at least to a paper [1] of E. Amigues from 1880. However, his argument included a flaw, the justification of termwise differentiability.

The condition (1) is true, e.g., for the functions \sin , \cos and \exp so we obtained the classical Taylor series representations that are usually derived in a standard calculus course. The above proof avoids the investigation of any form of the remainder but uses some multivariable calculus instead. It might be included in a multivariable calculus or differential equations course as a surprising application to Taylor series.

A single-variable variant

It is not so difficult to rephrase the above arguments as a single-variable proof of the Theorem. Let $y \in (a, b)$ be fixed and define the function

$$g(x) = f(x) + f'(x)(y - x) + \frac{f''(x)}{2!}(y - x)^2 + \frac{f'''(x)}{3!}(y - x)^3 + \dots$$

for $x \in (a, b)$. Then the series of g is absolutely and uniformly convergent due to similar reasons as before. By formal termwise differentiation we have

$$g'(x) = f'(x) + (f''(x)(y - x) - f'(x)) +$$

$$+ \left(\frac{f'''(x)}{2!}(y-x)^2 - f''(x)(y-x) \right) + \dots$$

The right-hand side is a telescoping series whose n th partial sum is

$$S_n(x) = \frac{f^{(n)}(x)}{(n-1)!}(y-x)^{n-1}.$$

The boundedness of the derivatives of f imply

$$|S_n(x)| \leq M \frac{(b-a)^{n-1}}{(n-1)!} \rightarrow 0 \text{ as } n \rightarrow \infty$$

which justifies the termwise differentiability of g . Moreover, $g'(x) = 0$ for $x \in (a, b)$, thus g is a constant function. Since $g(y) = f(y)$, it follows that

$$f(y) = f(x) + f'(x)(y-x) + \frac{f''(x)}{2!}(y-x)^2 + \frac{f'''(x)}{3!}(y-x)^3 + \dots$$

Finally, the choice $y = x + h$ yields the formula of the Theorem.

Summary. The phenomenon of traveling waves on a rope and the notion of Taylor series are well-known. Surprisingly, these two seemingly distant concepts have something in common. This connection can be used to derive the Taylor series representation for some real functions.

References

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2. L. C. Evans, *Partial Differential Equations*, AMS, Providence, 1998.
3. W. Rudin, *Principles of Mathematical Analysis*, 3rd ed., McGrawHill, 1976.