STABILIZATION OF SOLUTIONS TO A NONLINEAR SYSTEM MODELLING FLUID FLOW IN POROUS MEDIA

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1. Introduction

In this paper we investigate a system of nonlinear equations that models fluid flow through porous medium. A porous medium, roughly speaking, is a solid medium with lots of tiny holes, for example limestone. Such medium consits of two parts, the solid matrix and the holes. If the fluid carries dissolved chemical species, a variety of chemical reactions can occur that can change the porosity. This process was modelled by J. Logan, M. R. Petersen, T. S. Shores in [10] by the following system of equations in one dimension:

$$\omega(t,x)u_t(t,x) = \tag{1}$$
$$= \alpha \cdot (|v(t,x)|u_x(t,x))_x + K(\omega(t,x))p_x(t,x)u_x(t,x) - ku(t,x)g(\omega(t,x))$$

$$\omega_t(t,x) = bu(t,x)g(\omega(t,x)) \tag{2}$$

$$(K(\omega(t,x))p_x(t,x))_x = bu(t,x)g(\omega(t,x)), \tag{3}$$

$$v(t,x) = -K(\omega(t,x))p_x(t,x), \qquad t>0,\ x\in(0,1), \tag{4}$$

with initial and boundary conditions

$$u(0,x)=u_0(x), \quad \omega(0,x)=\omega_0(x) \quad x\in(0,1),$$
$$u(t,0)=u_1(t), \quad u_x(t,1)=0 \quad t>0,$$
$$p(t,0)=1, \quad p(t,1)=0 \quad t>0$$

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where ω is the porosity, u is the concentration of the dissolved solute, p is the pressure, v is the velocity, further, α, k, b are given constants, K and g are given real functions. For the details of making this model and on flow in such media see [7, 10] and the references there. In what follows, we investigate the following system of nonlinear differential equations:

$$D_t\omega(t,x) = f(t,x,\omega(t,x),u(t,x)), \qquad \omega(0,x) = \omega_0(x), \tag{5}$$

$$\begin{aligned} D_t u(t,x) - \sum_{i=1}^{n} D_i\left[a_i(t,x,\omega(t,x),u(t,x),Du(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x))\right] + \\ + a_0(t,x,\omega(t,x),u(t,x),Du(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x)) = \\ = g(t,x,\omega(t,x)), \qquad u(0,x) = 0, \end{aligned} \tag{6}$$

$$\begin{aligned} -\sum_{i=1}^{n} D_i[b_i(t,x,\omega(t,x),u(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x))] + \\ + b_0(t,x,\omega(t,x),u(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x)) = h(t,x,\omega(t,x),u(t,x)) \end{aligned} \tag{7}$$

with boundary conditions homogenous Dirichlet or Neumann, for example

$$\sum_{i=1}^{n} a_i(t,x,\omega(t,x),u(t,x),Du(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x))\nu_i = 0,$$

$$\mathbf{p}(t,x) = 0 \quad x \in \partial\Omega, t > 0,$$

where ν is the unit normal along the boundary. (The variable $\mathbf{p}$ is written by boldface letter for the purpose of distinguishing it from exponents p_1, p_2). Moreover, if $\partial\Omega = S_1 \cup S_2$ where $S_1 \cap S_2 = \emptyset$, then we can pose different boundary conditions on the elements of the partition. That is the case in the model (1)–(4) where the partitions are the endpoints of the interval $[0,1]$. Clearly, we can assume the boundary conditions to be homogeneous by substracting a suitable function from the unknown function.

The above system was investigated in [4] and it was shown there that due to some assumptions (that will be introduced later in this paper) the solution ω of (2) is positive thus we can divide equation (1) by ω. Hence the above system is a generalization of the model (1)–(4). Observe that for fixed u equation (5) is an ordinary differential equation with respect to the function ω; for fixed ω and p equation (6) is a parabolic equation with respect to the function u; and for fixed ω and u equation (7) is an elliptic equation with respect to the function $\mathbf{p}$. This shows that the above system is a hybrid evolutionary/elliptic problem, thus theorems of „usual" systems on partial differential equations do not work. In [4] existence of weak solutions to the above system were proved in $(0, T) \times \Omega$ (where $0 < T < \infty$) by using the theory of operators of monotone type. In the following we consider the equations in $(0, \infty) \times \Omega$. First we define the weak forms then we prove (under suitable conditions) existence of weak solutions, the boundedness of the solutions in appropriate norms in the time interval $(0, \infty)$, further, we show the stabilization of the solutions, in the sense that they converge to the stationary solutions as $t \to \infty$. We use some results and arguments of [11, 12, 13]. Finally, we give some examples.

2. Notation

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the uniform C^1 regularity property (see [1]), further, $2 \le p_1, p_2 < \infty$ be real numbers. For $0 < T < \infty$ let $Q_T := (0, T) \times \Omega$, and let $Q_\infty := (0, \infty) \times \Omega$. Denote by $W^{1,p_i}(\Omega)$ the usual Sobolev space with the norm

$$\|v\|_{W^{1,p_i}(\Omega)} = \left(\int_\Omega (|v|^{p_i} + \sum_{j=1}^{n} |D_j v|^{p_i}) \right)^{\frac{1}{p_i}}$$

where D_j denotes the distributional derivative with respect to the j-th variable (later we use the notation $D = (D_1, \dots, D_n)$). In addition, let V_i be a closed linear subspace of the space $W^{1,p_i}(\Omega)$ which contains $W_0^{1,p_i}(\Omega)$ (the closure of $C_0^\infty(\Omega)$ in $W^{1,p_i}(\Omega)$), and let $L^{p_i}(0, T; V_i)$ be the Banach space of measurable functions $u: (0, T) \to V_i$ such that $\|u\|_{V_i}^{p_i}$ is integrable and the norm is given by

$$\|u\|_{L^{p_i}(0,T;V_i)} = \left(\int_0^T \|u(t)\|_{V_i}^{p_i} dt \right)^{\frac{1}{p_i}}.$$

The dual space of $L^{p_i}(0, T; V_i)$ is $L^{q_i}(0, T; V_i^*)$ where $\frac{1}{p_i} + \frac{1}{q_i} = 1$ and V_i^* is the dual space of V_i. In what follows, we use the notation $X_i^T :=$ $= L^{p_i}(0, T; V_i)$. The pairing between V_i^* and V_i is denoted by $\langle \cdot, \cdot \rangle$, the pairing between $L^{q_i}(0, T; V_i^*)$ and $L^{p_i}(0, T; V_i)$ is denoted by $[\cdot, \cdot]$, further, $D_t u$ stands for the distribusional derivative (with respect to the variable t)

of a function $u \in L^{p_i}(0,T;V_i)$. It is well known (see [14]) that if $u \in L^{p_i}(0,T,V_i)$, $D_t u \in L^{q_i}(0,T;V_i^*)$ then $u \in C([0,T],L^2(\Omega))$ so that $u(0)$ makes sense. Denote by $X_i = L^{p_i}_{\text{loc}}(0,\infty;V_i)$ the space of measurable functions $u\colon(0,\infty)\to V_i$ such that for every $0<T<\infty$, $u|_{(0,T)} \in L^{p_i}(0,T;V_i)$. If for every $0<T<\infty$ there exists $D_t(u|_{(0,T)}) \in L^{q_i}(0,T;V_i^*)$ then $D_t u \in X_i^* = L^{q_i}_{\text{loc}}(0,\infty;V_i^*)$. Finally, let $L^\infty_{\text{loc}}(Q_\infty)$ denote the space of functions $\omega\colon Q_\infty \to \mathbb{R}$ such that $\omega|_{Q_T} \in L^\infty(Q_T)$ for every $0<T<\infty$.

3. The problem in $(0,\infty)$

3.1. Assumptions

We formulate some assumptions in order to prove existence of weak solutions. In the following ξ, (ζ_0,ζ), (η_0,η) refer for variables ω, (u,Du), $(\mathbf{p},D\mathbf{p})$, respectively.

A1. Functions $a_i\colon Q_\infty\times\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}\to\mathbb{R}$ $(i=0,\dots,n)$ have the Carathéodory property, i.e., they are measurable in $(t,x)\in Q_\infty$ for every $(\xi,\zeta_0,\zeta,\eta_0,\eta)\in\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ and continuous in $(\xi,\zeta_0,\zeta,\eta_0,\eta)\in \mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ for a.a. $(t,x)\in Q_\infty$.

A2. There exist a continuous function $c_1\colon\mathbb{R}\to\mathbb{R}^+$ and a function $k_1\in L^{q_1}(\Omega)$ such that

$$|a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta)| \le$$
$$\le c_1(\xi)\left(|\zeta_0|^{p_1-1}+|\zeta|^{p_1-1}+|\eta_0|^{\frac{p_2}{q_1}}+|\eta|^{\frac{p_2}{q_1}}+k_1(x)\right),$$

for a.a. $(t,x)\in Q_\infty$ and every $(\xi,\zeta_0,\zeta,\eta_0,\eta)\in\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ $(i=0,\dots,n)$.

A3. There exists a constant $C>0$ such that for a.a. $(t,x)\in Q_\infty$ and every $(\xi,\zeta_0,\zeta,\eta_0,\eta),(\xi,\zeta_0,\tilde\zeta,\eta_0,\eta)\in\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$

$$\sum_{i=1}^{n}\left(a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta)-a_i(t,x,\xi,\zeta_0,\tilde\zeta,\eta_0,\eta)\right)(\zeta_i-\tilde\zeta_i)\ge$$
$$\ge C\cdot|\zeta-\tilde\zeta|^{p_1}.$$

A4. There exist a constant $c_2 > 0$, a continuous function $\gamma\colon \mathbb{R} \to \mathbb{R}$ and a function $k_2 \in L^1(\Omega)$ such that

$$\sum_{i=0}^{n} a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta)\zeta_i \geq c_2\left(|\zeta_0|^{p_1} + |\zeta|^{p_1}\right) - \gamma(\xi)k_2(x)$$

for a.a. $(t,x) \in Q_\infty$ and every $(\xi,\zeta_0,\zeta,\eta_0,\eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$.

B1. Functions $b_i\colon Q_\infty \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1} \to \mathbb{R}$ $(i = 0,\dots,n)$ have the Carathéodory property, i.e., they are measurable in $(t,x) \in Q_\infty$ for every $(\xi,\zeta_0,\eta_0,\eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$ and continuous in $(\xi,\zeta_0,\eta_0,\eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$ for a.a. $(t,x) \in Q_\infty$.

B2. There exist a continuous function $\hat{c}_1\colon \mathbb{R} \to \mathbb{R}^+$ and a function $\hat{k}_1 \in L^{q_2}(\Omega)$ such that

$$|b_i(t,x,\xi,\zeta_0,\eta_0,\eta)| \leq \hat{c}_1(\xi)\left(|\eta_0|^{p_2-1} + |\eta|^{p_2-1} + |\zeta_0|^{\frac{p_1}{q_2}} + \hat{k}_1(x)\right)$$

for a.a. $(t,x) \in Q_\infty$ and every $(\xi,\zeta_0,\eta_0,\eta) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^{n+1}$ $(i = 0,\dots,n)$.

B3. There exists a constant $\hat{C} > 0$ such that for a.a. $(t,x) \in Q_\infty$ and every $(\xi,\zeta_0,\eta_0,\eta), (\xi,\zeta_0,\tilde{\eta}_0,\tilde{\eta}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$

$$\sum_{i=0}^{n}\left(b_i(t,x,\xi,\zeta_0,\eta_0,\eta) - b_i(t,x,\xi,\zeta_0,\tilde{\eta}_0,\tilde{\eta})\right)(\eta_i - \tilde{\eta}_i) \geq$$
$$\geq \hat{C}\cdot\left(|\eta_0 - \tilde{\eta}_0|^{p_2} + |\eta - \tilde{\eta}|^{p_2}\right).$$

B4. There exist a constant $\hat{c}_2 > 0$, a continuous function $\hat{\gamma}\colon \mathbb{R} \to \mathbb{R}$ and a function $\hat{k}_2 \in L^1(\Omega)$ such that

$$\sum_{i=0}^{n} b_i(t,x,\xi,\zeta_0,\eta_0,\eta)\eta_i \geq \hat{c}_2\left(|\eta_0|^{p_2} + |\eta|^{p_2}\right) - \hat{\gamma}(\xi)\left(|\zeta_0|^{p_1} + \hat{k}_2(x)\right)$$

for a.a. $(t,x) \in Q_\infty$ and every $(\xi,\zeta_0,\eta_0,\eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$.

F1. Function $f\colon Q_\infty \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function, i.e., it is measurable in $(t,x) \in Q_\infty$ for every fixed $(\xi,\zeta_0) \in \mathbb{R}^2$ and continuous in $(\xi,\zeta_0) \in \mathbb{R}^2$ for a.a. $(t,x) \in Q_\infty$. Further, for every bounded set $I \subset \mathbb{R}$ there exists a continuous function $K_1\colon \mathbb{R} \to \mathbb{R}^+$ such that

(i) $|K_1(\zeta_0)| \leq d_1|\zeta_0|^{\frac{p_1}{q_2}} + d_2$ for every $\zeta_0 \in \mathbb{R}$, with some nonnegative constants d_1, d_2 (depending on I),

(ii) for a.a. $(t,x) \in Q_\infty$ and every $(\xi,\zeta_0), (\tilde{\xi},\zeta_0) \in I \times \mathbb{R}$

$$|f(t,x,\xi,\zeta_0) - f(t,x,\tilde{\xi},\zeta_0)| \le K_1(\zeta_0) \cdot |\xi - \tilde{\xi}|.$$

F2. There exists a continuous function $K_2 \colon \mathbb{R} \to \mathbb{R}^+$ such that for a.a. $(t,x) \in$ $\in Q_\infty$ and all (ξ,ζ_0), $(\xi,\tilde{\zeta}_0) \in \mathbb{R}^2$

$$|f(t,x,\xi,\zeta_0) - f(t,x,\xi,\tilde{\zeta}_0)| \le K_2(\xi) \cdot |\zeta_0 - \tilde{\zeta}_0|.$$

F3. There exists $\omega^* \in L^\infty(\Omega)$ such that for a.a. $(t,x) \in Q_\infty$ and every $(\xi,\zeta_0) \in \mathbb{R}^2$, $(\xi - \omega^*(x)) \cdot f(t,x,\xi,\zeta_0) \le 0$.

G1. Function $g \colon Q_\infty \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, i.e., it is measurable in $(t,x) \in Q_\infty$ for every $\xi \in \mathbb{R}$ and continuous in $\xi \in \mathbb{R}$ for a.a. $(t,x) \in Q_\infty$.

G2. There exist a continuous function $c_3 \colon \mathbb{R} \to \mathbb{R}^+$ and a function $k_3 \in L^{q_1}(\Omega)$ such that

$$|g(t,x,\xi)| \le c_3(\xi) k_3(x)$$

for a.a. $(t,x) \in Q_\infty$ and every $\xi \in \mathbb{R}$.

H1. Function $h \colon Q_\infty \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function, i.e., it is measurable in $(t,x) \in Q_\infty$ for every $(\xi,\zeta_0) \in \mathbb{R}^2$ and continuous in $(\xi,\zeta_0) \in \mathbb{R}^2$ for a.a. $(t,x) \in Q_\infty$.

H2. There exist a continuous function $c_4 \colon \mathbb{R} \to \mathbb{R}^+$ and a function $k_4 \in L^{q_2}(\Omega)$ such that

$$|h(t,x,\xi,\zeta_0)| \le c_4(\xi) \left(|\zeta_0|^{\frac{p_1}{q_2}} + k_4(x) \right)$$

for a.a. $(t,x) \in Q_\infty$ and every $(\xi,\zeta_0) \in \mathbb{R}^2$.

3.2. Weak form

If the above assumptions are satisfied, we may define the weak form of the system (5)–(7) in Q_∞. First for arbitrary

$$\omega \in L^\infty_{\text{loc}}(Q_\infty),\ u \in L^{p_1}_{\text{loc}}(0,\infty; V_1),\ \mathbf{p} \in L^{p_2}_{\text{loc}}(0,\infty; V_2) \text{ and } t \in (0,\infty)$$

define functionals

$$[A(\omega,u,p)](t) \in V_1^*, [B(\omega,u,\mathbf{p})](t) \in V_2^*,$$
$$[G(\omega)](t) \in V_1^*, [H(\omega,u)](t) \in V_2^*$$

by

$$\langle [A(\omega, u, p)](t), v_1 \rangle :=$$

$$= \int_\Omega \sum_{i=1}^n a_i(t, x, \omega(t,x), u(t,x), Du(t,x), \mathbf{p}(t,x), D\mathbf{p}(t,x)) D_i v_1(x)\, dx +$$

$$+ \int_\Omega a_0(t, x, \omega(t,x), u(t,x), Du(t,x), \mathbf{p}(t,x), D\mathbf{p}(t,x)) v_1(x)\, dx,$$

$$\langle [B(\omega, u, \mathbf{p})](t), v_2 \rangle :=$$

$$= \int_\Omega \sum_{i=1}^n b_i(t, x, \omega(t,x), u(t,x), \mathbf{p}(t,x), D\mathbf{p}(t,x)) D_i v_2(x)\, dx +$$

$$+ \int_\Omega b_0(t, x, \omega(t,x), u(t,x), \mathbf{p}(t,x), D\mathbf{p}(t,x)) v_2(x)\, dx,$$

$$\langle [G(\omega)](t), v_1 \rangle := \int_\Omega g(t, x, \omega(t,x)) v_1(x)\, dx,$$

$$\langle [H(\omega, u)](t), v_2 \rangle := \int_\Omega h(t, x, \omega(t,x), u(t,x)) v_2(x)\, dx \tag{8}$$

for all $v_1 \in V_1$ and $v_2 \in V_2$. Further, for every $0 < T < \infty$ we define operators

$$A_T: L^\infty(Q_T) \times X_1^T \times X_2^T \to (X_1^T)^*, \quad B_T: L^\infty(Q_T) \times X_1^T \times X_2^T \to (X_2^T)^*,$$
$$G_T: L^\infty(Q_T) \to (X_1^T)^*, \quad H_T: L^\infty(Q_T) \times X_1^T \to (X_2^T)^*$$

such that for all $w_1 \in X_1^T$ and $w_2 \in X_2^T$

$$\begin{aligned}
&[A_T(\omega, u, \mathbf{p}), w_1] = \int_0^T \langle [A(\omega, u, \mathbf{p})](t), w_1(t) \rangle\, dt,\\
&[B_T(\omega, u, \mathbf{p}), w_2] = \int_0^T \langle [B(\omega, u, \mathbf{p})](t), w_2(t) \rangle\, dt,\\
&[G_T(\omega), w_1] = \int_0^T \langle [G(\omega)](t), w_1(t) \rangle\, dt,\\
&[H_T(\omega, u), w_2] = \int_0^T \langle [H(\omega, u)](t), w_2(t) \rangle\, dt.
\end{aligned} \tag{9}$$

In addition, let us introduce the linear operators $L_T : D(L_T) \to (X_1^T)^*$ by the formula

$$D(L_T) = \{u \in X_1^T : D_t u \in (X_1^T)^*,\ u(0) = 0\}, \qquad L_T u = D_t u.$$

Now we are ready to define the weak form of (5)–(7), namely, we say that $\omega \in L^\infty_{\text{loc}}(Q_\infty)$, $u \in L^{p_1}_{\text{loc}}(0,\infty; V_1)$, $\mathbf{p} \in L^{p_2}_{\text{loc}}(0,\infty; V_2)$ are weak solutions of the system (5)–(7) if for every $0 < T < \infty$

$$\omega(t,x) = \omega_0(x) + \int_0^t f(s,x,\omega(s,x),u(s,x))\,ds, \quad \text{a.e. } (t,x) \in Q_T \tag{10}$$

$$L_T(u|_{Q_T}) + A_T(\omega|_{Q_T}, u|_{Q_T}, \mathbf{p}|_{Q_T}) = G_T(\omega|_{Q_T}) \tag{11}$$

$$B_T(\omega|_{Q_T}, u|_{Q_T}, \mathbf{p}|_{Q_T}) = H_T(\omega|_{Q_T}, u|_{Q_T}). \tag{12}$$

It is well known (see e.g. [9]) that one gets the above weak forms by considering sufficiently smooth solutions and then using Green's theorem, after that one considers the equations in the spaces X_i. It is clear that if the boundary condition is homogenous Neumann than $V_i = W^{1,p_i}(\Omega)$ (since the boundary term vanishes in Green's theorem) and if we have homogeneous Dirichlet boundary condition then $V_i = W_0^{1,p_i}(\Omega)$ (in order to eliminate the boundary terms in Green's theorem). Further, if we have a partition then for example in our one dimensional equation (1) with homogenous boundary conditions $V_1 = \{v \in W^{1,p_1}(0,1) : v(t,0) = 0\}$, and in addition $V_2 = W_0^{1,p_2}(0,1)$. In the next section we prove that the earlier introduced assumptions imply existence of solutions of the above system.

3.3. Existence of weak solutions

THEOREM 1. *Suppose that conditions A1–A4, B1–B4, F1–F3, G1–G2, H1–H2 are satisfied. Then for all* $\omega_0 \in L^\infty(\Omega)$ *there exist solutions* $\omega \in L^\infty(Q_\infty)$, $u \in L^{p_1}_{\text{loc}}(0,\infty; V_1)$, $\mathbf{p} \in L^{p_2}_{\text{loc}}(0,\infty; V_2)$ *of problem* (10)–(12).

PROOF. The main idea is the following. By [4] for every $0 < T < \infty$ there exist solutions in Q_T. By using a diagonal process we can choose weakly convergent subsequences from the solutions and then prove that the limit functions are solutions in Q_∞.

Let (T_k) be a monotone increasing sequence of positive numbers such that $T_k \to +\infty$. Then by Theorem 2.1 in [4], for every T_k we have $\omega_k \in \in L^\infty(Q_{T_k})$, $u_k \in L^{p_1}(0, T_k; V_1)$, $\mathbf{p}_k \in L^{p_2}(0, T_k; V_2)$, such that

$$\omega_k(t,x) = \omega_0(x) + \int_0^t f(s,x,\omega_k(s,x),u_k(s,x))\,ds, \quad (t,x) \in Q_{T_k} \tag{13}$$

$$L_{T_k} u_k + A_{T_k}(\omega_k, u_k, \mathbf{p}_k) = G_{T_k}(\omega_k) \tag{14}$$

$$B_{T_k}(\omega_k, u_k, \mathbf{p}_k) = H_{T_k}(\omega_k, u_k). \tag{15}$$

(We omit the notation $|_{Q_{T_m}}$ if it is not confusing, since the operators and the norms contain the information about the space). By similar arguments as in Proposition 2.2 of the cited paper, it is easy to see that

$$\|\omega_k\|_{L^\infty(Q_{T_m})} \leq \|\omega_0\|_{L^\infty(\Omega)} + \|\omega^*\|_{L^\infty(\Omega)}. \tag{16}$$

Now by following the proof of Theorem 2.1 in [4] word for word and by using the boundedness of the sequence (ω_k) in $L^\infty(Q_{T_m})$ for all m one obtains that for fixed $m \in \mathbb{N}$, $(u_k|_{Q_{T_m}})$, $(L_{T_m} u_k|_{Q_{T_m}})$ and $(\mathbf{p}_k|_{Q_{T_m}})$ are bounded in $X_1^{T_m}$, $(X_1^{T_m})^*$ and $X_2^{T_m}$, respectively.

Now let $m = 1$. Since $(u_k), (L_{T_1} u_k), (\mathbf{p}_k)$ are bounded sequences in reflexive Banach spaces $X_1^{T_1}, (X_1^{T_1})^*, X_2^{T_1}$, respectively, there exist subsequences $(u_{1,k}) \subset (u_k), (\mathbf{p}_{1,k}) \subset (\mathbf{p}_k)$ and functions $u_{1,*} \in X_1^{T_1}$, $\mathbf{p}_{1,*} \in X_2^{T_1}$ such that

$$u_{1,k} \to u_{1,*} \text{ weakly in } X_1^{T_1},$$

$$L_{T_1} u_{1,k} \to L_{T_1} u_{1,*} \text{ weakly in } (X_1^{T_1})^*,$$

$$\mathbf{p}_{1,k} \to \mathbf{p}_{1,*} \text{ weakly in } X_2^{T_1}.$$

If $(u_{m-1,k})_{k \geq m-1}$ is given then

$$(u_{m-1,k})_{k \geq m-1},\ (L_{T_{m-1}} u_{m-1,k})_{k \geq m-1},\ (\mathbf{p}_{m-1,k})_{k \geq m-1}$$

are bounded in reflexive spaces $X_1^{T_{m-1}}, (X_1^{T_{m-1}})^*, X_2^{T_{m-1}}$, thus there exist subsequences $(u_{m,k}) \subset (u_{m-1,k}), (\mathbf{p}_{m,k}) \subset (\mathbf{p}_{m-1,k})$ and functions $u_{m,*} \in \in X_1^{T_m}, \mathbf{p}_{m,*} \in X_2^{T_m}$ such that

$$u_{m,k} \to u_{m,*} \text{ weakly in } X_1^{T_m},$$

$$L_{T_m} u_{m,k} \to L_{T_m} u_{m,*} \text{ weakly in } (X_1^{T_m})^*,$$

$$\mathbf{p}_{m,k} \to \mathbf{p}_{m,*} \text{ weakly in } X_2^{T_m}.$$

It is easy to see that for each fixed $l < m$ the above weak convergences hold in $X_1^{T_l}$, $(X_1^{T_l})^*$, $X_2^{T_l}$, respectively, which yields $u_{m,*}|_{Q_{T_l}} = u_{l,*}$ and $\mathbf{p}_{m,*}|_{Q_{T_l}} = u_{l,*}$ for $l < m$. Consequently there exist unique functions $u\colon (0,\infty) \to V_1, \mathbf{p}\colon (0,\infty) \to V_2$ such that $u|_{Q_{T_m}} = u_{m,*}, \mathbf{p}|_{Q_{T_m}} = \mathbf{p}_{m,*}$ and $u_{m,*} \in D(L_{T_m})$ for every $m \in \mathbb{N}$. This means that $u \in L^{p_1}_{\mathrm{loc}}(0,\infty;V_1)$ and $\mathbf{p} \in L^{p_2}_{\mathrm{loc}}(0,\infty;V_2)$. Consider the sequences $(u_k) = (u_{k,k}), (\mathbf{p}_k) = (\mathbf{p}_{k,k})$ and the appropriate sequence (ω_k). Observe that $u_k \to u$ weakly in $X_1^{T_m}$, $\mathbf{p}_k \to \mathbf{p}$ weakly in $X_2^{T_m}$. Thus by the well known embedding theorem (see [9]) we may assume that $u_k \to u$ in $L^{p_1}(Q_{T_m})$. Then from Propositions 2.2, 2.3 in [4] it follows that for every m there exists $\omega_{m,*} \in L^\infty(Q_{T_m})$ such that $(\omega_k) \to \omega_{m,*}$ a.e. in Q_{T_m} and

$$\omega_{m,*}(t,x) = \omega_0(x) + \int_0^t f(s,x,\omega_{m,*}(s,x),u_{m,*}(s,x))\,ds, \quad (t,x) \in Q_{T_m}.$$

Since for every fixed $u_{m,*}$ the solution of the above equation is unique, further, functions $(u_{m,*})$ are the restrictions of the function u to Q_{T_m}, it follows that there exists a unique $\omega \in L^\infty_{\mathrm{loc}}(Q_\infty)$ such that $\omega_{m,*} = \omega|_{Q_{T_m}}$ for every m and

$$\omega(t,x) = \omega_0(x) + \int_0^t f(s,x,\omega(s,x),u(s,x))\,ds \quad (t,x) \in Q_\infty.$$

By similar arguments as in Proposition 2.2 in [4] it is easy to see that $\omega \in L^\infty(Q_\infty)$. Now fix $m \in \mathbb{N}$. We conclude from the above arguments that

$$\omega_k \to \omega \quad \text{a.e. in} \quad Q_{T_m}$$
$$u_k \to u \quad \text{weakly in} \quad X_1^{T_m}, \text{ strongly in } L^{p_1}(Q_{T_m}), \text{ a.e. in } Q_{T_m};$$
$$L_{T_m} u_k \to L_{T_m} u \quad \text{weakly in} \quad (X_1^{T_m})^*;$$
$$\mathbf{p}_k \to \mathbf{p} \quad \text{weakly in} \quad X_2^{T_m}.$$

By applying word for word step 3 of the proof of Theorem 2.1 in [4], by the above convergences it is not difficult to show that (for a suitable subsequence) $u_k \to u$ strongly in $X_1^{T_m}$, $\mathbf{p}_k \to \mathbf{p}_{m,*}$ strongly in $X_2^{T_m}$ and

$$L_{T_m} u|_{Q_{T_m}} + A_{T_m}(\omega|_{Q_{T_m}}, u|_{Q_{T_m}}, \mathbf{p}|_{Q_{T_m}}) = G_{T_m}(\omega|_{Q_{T_m}})$$
$$B_{T_m}(\omega|_{Q_{T_m}}, u|_{Q_{T_m}}, \mathbf{p}|_{Q_{T_m}}) = H_{T_m}(\omega|_{Q_{T_m}}, u|_{Q_{T_m}}).$$

This means that $\omega, u, \mathbf{p}$ are solutions in $(0, \infty)$, so the proof of the theorem is complete. ∎

4. Boundedness of solutions

In this section we show that under some further assumptions, the solutions of (10)–(12) are bounded in appropriate norms in the time interval $(0, \infty)$. First suppose the following.

B4*. There exist a constant $\hat{c}_2 > 0$, a continuous function $\hat{\gamma}: \mathbb{R} \to \mathbb{R}$ and a function $\hat{k}_2 \in L^1(\Omega)$ such that

$$\sum_{i=0}^{n} b_i(t, x, \xi, \zeta_0, \eta_0, \eta)\eta_i \geq \hat{c}_2 \left(|\eta_0|^{p_2} + |\eta|^{p_2}\right) - \hat{\gamma}(\xi) \left(|\zeta_0|^2 + \hat{k}_2(x)\right)$$

for a.a. $(t, x) \in Q_\infty$ and every $(\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$.

H2*. There exist a continuous function $c_4: \mathbb{R} \to \mathbb{R}^+$ and a function $k_4 \in L^{q_2}(\Omega)$ such that

$$|h(t, x, \xi, \zeta_0)| \leq c_4(\xi) \left(|\zeta_0|^{\frac{2}{q_2}} + k_4(x)\right)$$

for a.a. $(t, x) \in Q_\infty$ and every $(\xi, \zeta_0) \in \mathbb{R}^2$.

THEOREM 2. *Assume that conditions A1–A4, B1–B3, B4*, F1–F3, G1–G2, H1, H2* are fulfilled and let $\omega, u, \mathbf{p}$ be solutions of* (10)–(12). *Then $\omega \in L^\infty(Q_\infty)$, $u \in L^\infty(0, \infty; L^2(\Omega))$, $\mathbf{p} \in L^\infty(0, \infty; V_2)$.*

PROOF. In Theorem 1 we have verified that $\omega \in L^\infty(Q_\infty)$ (which was a trivial consequence of (16)). In the following let $y(t) = \|u(t, \cdot)\|^2_{L^2(\Omega)}$. First note that $u \in C([0, T], L^2(\Omega))$ thus y is continuous in $[0, T]$ (see, e.g., [14]). We show that y is bounded in $(0, \infty)$. Since u is a solution of (11) for all $0 < T < \infty$, thus for arbitrary $0 < T_1 < T_2 < \infty$ we have

$$\int_{T_1}^{T_2} \langle D_t u(t), u(t)\rangle \, dt + \int_{T_1}^{T_2} \langle [A(\omega, u, \mathbf{p})](t), u(t)\rangle \, dt =$$

$$= \int_{T_1}^{T_2} \langle [G(\omega)](t), u(t)\rangle \, dt. \tag{17}$$

By $\omega \in L^{\infty}(Q_{\infty})$ and G2, $\|[G(\omega)](t)\|_{V_1^*}$ is bounded, thus by Young's inequality we obtain for $\varepsilon > 0$

$$\int_{T_1}^{T_2} \langle [G(\omega)](t), u(t)\rangle \, dt \leq$$

$$\leq \int_{T_1}^{T_2} \left(\frac{\varepsilon^{p_1}}{p_1} \|u(t)\|_{V_1}^{p_1} + \frac{1}{q_1 \varepsilon^{q_1}} \|[G(\omega)](t)\|_{V_1^*} \right) dt \leq$$

$$\leq \int_{T_1}^{T_2} \left(\frac{\varepsilon^{p_1}}{p_1} \|u(t)\|_{V_1}^{p_1} + c(\varepsilon) \right) \, dt.$$

By using the relation $\int_{T_1}^{T_2} \langle D_t u(t), u(t)\rangle \, dt = y(T_2) - y(T_1)$ (see [14]) and condition A4 on the left hand side of (17), further, by applying the above estimates with sufficiently small ε on the right hand side it follows

$$\frac{1}{2} \left(y(T_1) - y(T_2) \right) + \frac{1}{2} c_2 \int_{T_1}^{T_2} \|u(t)\|_{V_1}^{p_1} \, dt \leq$$

$$\leq \text{const} \int_{T_1}^{T_2} \int_{\Omega} \left(|\gamma(\omega(t,x)) k_2(x)| + 1 \right) \, dx \, dt.$$

Since $\omega \in L^{\infty}(Q_{\infty})$ and γ is continuous, the integrand (with respect to variable t) is bounded, thus $\text{const} \cdot (T_2 - T_1)$ is an upper bound of the right hand side. By the continuous embedding $L^2(\Omega) \hookrightarrow V_1$,

$$\|u(t)\|_{L^2(\Omega)} \leq \text{const} \cdot \|u(t)\|_{V_1},$$

hence

$$y(T_2) - y(T_1) + \text{const} \int_{T_1}^{T_2} y(t)^{\frac{p_1}{2}} \, dt \leq \text{const} \cdot (T_2 - T_1). \tag{18}$$

Now we show that the above inequality implies the boundedness of y. Suppose the contrary. Then for every $M > 0$ there exist $0 < T_1 < T_2 < \infty$ such that $y(T_2) = M$ and $M - 1 \le y(t) \le M$ if $T_1 \le t \le T_2$. By choosing these T_1, T_2, from (18) we obtain

$$\text{const} \cdot (T_2 - T_1)(M - 1) \le \text{const} \cdot (T_2 - T_1)$$

for every $M > 0$. This is impossible, thus y is bounded.

It remains to show that $\mathbf{p} \in L^\infty(0, \infty; V_2)$. The proof goes the same way as the previous part (moreover it is simpler since there is no derivative with respect to t), from $B(\omega, u, \mathbf{p}) = H(\omega, u)$, by using conditions B4*, H2* and the boundedness of ω we obtain

$$\|p(t)\|_{V_2}^{p_2} \le \text{const} \left(\|u(t)\|^2_{L^2(\Omega)} + 1 \right).$$

In the previous part of the proof we have shown that $y(t) = \|u(t)\|^2_{L^2(\Omega)}$ is bounded thus the above inequality implies the desired $\mathbf{p} \in L^\infty(0, \infty; V_2)$. The proof of Theorem 2 is complete. ∎

5. Stabilization of solutions

In this section we consider a special case of the system (10)–(12), namely, let $p_1 = p_2 = p$ (thus $q_1 = q_2 = q$, $V_1 = V_2 = V$ and $X_1 = X_2 = X$), $h = f$ and consider the following problem: for every $0 < T < \infty$

$$\omega(t, x) = \omega_0(x) + \int_0^t f(s, x, \omega(s, x), u(s, x))\, ds, \quad (t, x) \in Q_T \tag{19}$$

$$L_T(u|_{Q_T}) + A_T(\omega|_{Q_T}, u|_{Q_T}, \mathbf{p}|_{Q_T}) = G_T(\omega|_{Q_T}) \tag{20}$$

$$B_T(\omega|_{Q_T}, u|_{Q_T}, \mathbf{p}|_{Q_T}) = F_T(\omega|_{Q_T}, u|_{Q_T}). \tag{21}$$

The above operator F_T is defined through formula (9) with $H = F$, where operator F is given by (8) with $h = f$. We note that in the original model (1)–(4) $h = f$ thus the above sytem is also the generalization of it. In what follows, we show stabilization of the solutions of the above sytem. That is, we prove the convergence (in some sense) of solutions as $t \to \infty$ to the solutions of the stationary system. We need some further assumptions:

A5. There exist Carathéodory functions $a_{i,\infty}: \Omega \times \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ ($i = 0, \dots, n$), i.e., they are measurable in $x \in \Omega$ for every $(\xi, \zeta_0, \zeta, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ and continuous in $(\xi, \zeta_0, \zeta, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$

for a.a. $x \in \Omega$. Further, for a.a. $x \in \Omega$ and every $(\zeta_0, \zeta, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$, $\xi^* \in \mathbb{R}$

$$\lim_{\substack{t\to\infty \\ \xi\to\xi^*}} a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta) = a_{i,\infty}(x,\xi^*,\zeta_0,\zeta,\eta_0,\eta).$$

B5. There exist Carathéodory functions $b_{i,\infty}\colon \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1} \to \mathbb{R}$ ($i = 0, \dots, n$), i.e., they are measurable in $x \in \Omega$ for every $(\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1}$ and continuous in $(\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \times \mathbb{R}^{n+1}$ for a.a. $x \in \Omega$. Further, for a.a. $x \in \Omega$ and every $(\zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1}$, $\xi^* \in \mathbb{R}$,

$$\lim_{\substack{t\to\infty \\ \xi\to\xi^*}} b_i(t,x,\xi,\zeta_0,\eta_0,\eta) = b_{i,\infty}(x,\xi^*,\zeta_0,\eta_0,\eta).$$

AB There exists a positive constant $\mathscr{C}$ such that for a.a. $(t,x) \in Q_\infty$, every $\xi \in \mathbb{R}$ and $(\zeta_0, \zeta, \eta_0, \eta)$, $(\tilde{\zeta}_0, \tilde{\zeta}, \tilde{\eta}_0, \tilde{\eta}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$

$$\sum_{i=0}^{n} \left(a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta) - a_i(t,x,\xi,\tilde{\zeta}_0,\tilde{\zeta},\tilde{\eta}_0,\tilde{\eta}) \right) (\zeta_i - \tilde{\zeta}_i) +$$
$$+ \sum_{i=0}^{n} \left(b_i(t,x,\xi,\zeta_0,\eta_0,\eta) - b_i(t,x,\xi,\tilde{\zeta}_0,\tilde{\eta}_0,\tilde{\eta}) \right) (\eta_i - \tilde{\eta}_i) \ge$$
$$\ge \mathscr{C} \cdot \left(|\zeta_0 - \tilde{\zeta}_0|^p + |\zeta - \tilde{\zeta}|^p + |\eta_0 - \tilde{\eta}_0|^p + |\eta - \tilde{\eta}|^p \right).$$

F1/(i*) Suppose (instead of condition F1/(i)) that there exist nonnegative constants d_1, d_2 such that $|K_1(\zeta_0)| \le d_1 |\zeta_0|^{\frac{2}{q}} + d_2$ for every $\zeta_0 \in \mathbb{R}$.

F4. There exists a positive constant m such that $(\xi - \omega^*(x)) f(t,x,\xi,\zeta_0) \le -m(\xi - \omega^*(x))^2$ for a.a. $(t,x) \in Q_\infty$ and every $(\xi, \zeta_0) \in \mathbb{R}^2$.

G3. There exists a Carathéodory function $g_\infty\colon \Omega \times \mathbb{R} \to \mathbb{R}$ such that for a.a. $x \in \Omega$ and every $\xi^* \in \mathbb{R}$,

$$\lim_{\substack{t\to\infty \\ \xi\to\xi^*}} g(t,x,\xi) = g_\infty(x,\xi^*).$$

Now introduce operators

$$A_\infty\colon L^\infty(\Omega) \times V \times V \to V^*,$$
$$B_\infty\colon L^\infty(\Omega) \times V \times V \to V^*,$$
$$G_\infty\colon L^\infty(\Omega) \to V^*$$

by

$$\langle A_\infty(\omega, u, \mathbf{p}), v\rangle :=$$
$$= \int_\Omega \sum_{i=1}^n a_{i,\infty}(x, \omega(x), u(x), Du(x), \mathbf{p}(x), D\mathbf{p}(x)) D_i v(x)\, dx +$$
$$+ \int_\Omega a_{0,\infty}(x, \omega(x), u(x), Du(x), \mathbf{p}(x), D\mathbf{p}(x)) v(x)\, dx,$$

$$\langle B_\infty(\omega, u, \mathbf{p}), v\rangle :=$$
$$= \int_\Omega \sum_{i=1}^n b_{i,\infty}(x, \omega(x), u(x), \mathbf{p}(x), D\mathbf{p}(x)) D_i v(x)\, dx +$$
$$+ \int_\Omega b_{0,\infty}(x, \omega(x), u(x), \mathbf{p}(x), D\mathbf{p}(x)) v(x)\, dx,$$

$$\langle G_\infty(\omega), v\rangle := \int_\Omega g_\infty(x, \omega(x)) v(x)\, dx,$$

where $v \in V$.

THEOREM 3. *Assume that conditions A1–A5, B1–B3, B4*, B5, AB, F1–F4, F1/(i*), G1–G3 are satisfied (with $p = p_1 = p_2$). Then there exist $u_\infty \in V$, $\mathbf{p}_\infty \in V$ such that for the solutions ω, u, $\mathbf{p}$ of problem* (19)–(21),

$$\omega(t,\cdot) \to \omega^* \text{ in } L^\infty(\Omega),\ u(t) \to u_\infty \text{ in } L^2(\Omega),\ \int_{t-1}^{t+1} \|u(s) - u_\infty\|_V^p\, ds \to 0,$$

$$\int_{t-1}^{t+1} \|\mathbf{p}(s) - \mathbf{p}_\infty\|_V^p\, ds \to 0,\ [F(\omega, u)](t) \to 0 \text{ in } V^*,\ [G(\omega)](t) \to G_\infty(\omega^*)$$

in V^* as $t \to \infty$, further,

$$A_\infty(\omega^*, u_\infty, \mathbf{p}_\infty) = G_\infty(\omega^*)$$
$$B_\infty(\omega^*, u_\infty, \mathbf{p}_\infty) = 0.$$

REMARK 4. Precisely, by the convergence $s(t) \to 0$ as $t \to \infty$ where $s: \mathbb{R}^+ \to M$ is a measurable function, and M is a normed space, we mean that for all $\varepsilon > 0$ there exists t_0 such that $\|s(t)\|_M \le \varepsilon$ for a.a. $t > t_0$.

PROOF. First we show that $\omega(t,\cdot) \to \omega^*$ in $L^\infty(\Omega)$ as $t \to \infty$. Fix $x \in \Omega$ and assume that $\omega_0(x) > \omega^*(x)$. By using similar arguments as in the proof of Proposition 2.5 in [4] we obtain that for $t > 0$, $\omega(t,x) > \omega^*(x)$. Then conditions F3 and F4 yield $f(t,x,\omega(t,x),u(t,x)) \le -m(\omega(t,x) - \omega^*(x))$. Since ω is absolutely continuous, it is a.e. differentiable, so

$$\omega'(t,x) = f(t,x,\omega(t,x),u(t,x)) \le -m(\omega(t,x) - \omega^*(x)).$$

By the positivity of $\omega - \omega^*$ we obtain

$$\frac{\omega'(t,x)}{\omega(t,x) - \omega^*(x)} \le -m$$

hence

$$\omega(t,x) - \omega^*(x) \le \omega_0(x)e^{-mt}.$$

When $\omega_0(x) < \omega^*(x)$ one has the estimate $-\omega_0(x)e^{-mt} \le \omega(t,x) - \omega^*(x) \ge$, ????
thus $\|\omega(t,\cdot) - \omega^*(\cdot)\|_{L^\infty(\Omega)} \le \|\omega_0\|_{L^\infty(\Omega)}e^{-mt}$. This means that $\omega(t,\cdot) \to$ $\to \omega^*(\cdot)$ in $L^\infty(\Omega)$ as $t \to \infty$. Note, that this implies that $\omega(t,x) \to \omega^*(x)$ for a.a. $x \in \Omega$ as $t \to \infty$.

Now we prove that $[F(\omega,u)](t) \to 0$ in V^* as $t \to \infty$. Clearly, by conditions F1, F1/(i*), F3

$$\begin{aligned}\|[F(\omega,u)](t)\|_{V^*}^q &\le \int_\Omega |f(t,x,\omega(t,x),u(t,x))|^q\,dx \\ &\le \int_\Omega |K_1(u(t,x))|^q |\omega(t,x) - \omega^*(x)|^q\,dx \\ &\le \text{const}\cdot\left(\|u(t)\|_{L^2(\Omega)}^2 + \text{const}\right)\cdot\left(\|\omega(t,\cdot) - \omega^*\|_{L^\infty(\Omega)}^q\right).\end{aligned}$$

The multiplicator of the term $\|\omega(t,\cdot) - \omega^*\|_{L^\infty(\Omega)}$ is bounded by the previous section, thus the right hand side of the above inequality tends to 0 as $t \to \infty$.

Now we verify that $[G(\omega)](t) \to G_\infty(\omega^*)$ in V^*. Observe that

$$\|G(\omega)](t) - G_\infty(\omega^*)\|_{V_2^*}^q \le \int_\Omega |g(t,x,\omega(t,x)) - g_\infty(x,\omega^*)|^q\,dx$$

and the integrand is a.e. convergent to 0 in Ω as $t \to \infty$ by G3. Further, by G2, G3, $|g_\infty(x,\omega)| \le c_3(\omega)k_3(x)$, hence $|g(t,x,\omega(t,x)) - g_\infty(x,\omega^*)|^q \le$ $\le \left(\|c_3(\omega)\|_{L^\infty(Q_\infty)} + \|c_3(\omega^*)\|_{L^\infty(\Omega)}\right)|k_3(x)|^q$ and the right hand side is integrable, thus Lebesgue's theorem implies that the above integral converges to 0 as $t \to \infty$.

Now we show that problem

$$A_\infty(\omega^*, u_\infty, \mathbf{p}_\infty) = G_\infty(\omega^*) \tag{22}$$

$$B_\infty(\omega^*, u_\infty, \mathbf{p}_\infty) = 0 \tag{23}$$

has a unique solution $u_\infty \in V, \mathbf{p}_\infty \in V$ (for fixed ω^*). First observe that functions $a_{i,\infty}, b_{i,\infty}, f, g$ satisfy conditions A1–A4, B1–B4, G1–G2, H1–H2 (with variables x instead of (t, x)). Then by the using the idea of the main theorem of [4] (successive approximation) it is easy to see that there exist solutions $u_\infty \in V, \mathbf{p}_\infty \in V$ of problem (22)–(23) (the proof goes almost word for word, but it is simpler since there is no ω in the above problem). Uniqueness follows from condition AB, indeed if $u_1, \mathbf{p}_1$ and $u_2, \mathbf{p}_2$ are solutions then

$$\begin{aligned} 0 = &\langle A_\infty(\omega^*, u_1, \mathbf{p}_1) - A_\infty(\omega^*, u_2, \mathbf{p}_2), u_1 - u_2\rangle + \\ &+ \langle B_\infty(\omega^*, u_1, \mathbf{p}_1) - B_\infty(\omega^*, u_2, \mathbf{p}_2), \mathbf{p}_1 - \mathbf{p}_2\rangle \geq \\ &\geq \mathscr{C} \cdot \left(\|u_1 - u_2\|_V^{p_1} + \|\mathbf{p}_1 - \mathbf{p}_2\|_V^p \right) \end{aligned}$$

which implies $u_1 = u_2$ and $\mathbf{p}_1 = \mathbf{p}_2$.

In order to show the desired convergences we prove inequality for u and p. From equations (19)–(21) and (22)–(23) we obtain

$$\begin{aligned} &\langle D_t(u(t) - u_\infty), u(t) - u_\infty\rangle + \langle [A(\omega, u, \mathbf{p})](t) - A_\infty(\omega^*, u_\infty, \mathbf{p}_\infty), u(t) - u_\infty\rangle + \\ &+ \langle [B(\omega, u, \mathbf{p})](t) - B_\infty(\omega^*, u_\infty, \mathbf{p}_\infty), \mathbf{p}(t) - \mathbf{p}_\infty\rangle = \\ &= \langle [G(\omega)](t) - G_\infty(\omega^*), u(t) - u_\infty\rangle + \langle [F(\omega, u)](t), \mathbf{p}(t) - \mathbf{p}_\infty\rangle. \end{aligned} \tag{24}$$

Observe that the first term on the left hand side of the above equation equals to $\frac{1}{2}y'(t)$ where $y(t) = \int_\Omega (u(t) - u_\infty)^2$ (note that y is bounded in $[0, \infty)$ by Theorem 2). Further, for the second and third terms of the above equation we have by condition AB and Young's inequality

$$\begin{aligned} &\langle [A(\omega, u, \mathbf{p})](t) - A_\infty(\omega^*, u_\infty, \mathbf{p}_\infty), u(t) - u_\infty\rangle + \\ &+ \langle [B(\omega, u, \mathbf{p})](t) - B_\infty(\omega^*, u_\infty, \mathbf{p}_\infty), \mathbf{p}(t) - \mathbf{p}_\infty\rangle = \\ &= \langle [A(\omega, u, \mathbf{p})](t) - [A(\omega, u_\infty, \mathbf{p}_\infty)](t), u(t) - u_\infty\rangle + \\ &+ \langle [B(\omega, u, \mathbf{p})](t) - [B(\omega, u_\infty, \mathbf{p}_\infty)](t), \mathbf{p}(t) - \mathbf{p}_\infty\rangle + \\ &+ \langle [A(\omega, u_\infty, \mathbf{p}_\infty)](t) - A_\infty(\omega^*, u_\infty, \mathbf{p}_\infty), u(t) - u_\infty\rangle + \\ &+ \langle [B(\omega, u_\infty, \mathbf{p}_\infty)](t) - B_\infty(\omega^*, u_\infty, \mathbf{p}_\infty), \mathbf{p}(t) - \mathbf{p}_\infty\rangle \geq \\ &\geq \mathscr{C} \cdot \left(\|u(t) - u_\infty\|_V^p + \|\mathbf{p}(t) - \mathbf{p}_\infty\|_V^p \right) - \frac{\varepsilon^p}{p} \|u(t) - u_\infty\|_V^p - \end{aligned}$$

$$-\frac{\varepsilon^p}{p}\|\mathbf{p}(t)-\mathbf{p}_\infty\|_V^p-\frac{1}{q\varepsilon^q}\|[A(\omega,u_\infty,\mathbf{p}_\infty)](t)-A_\infty(\omega^*,u_\infty,\mathbf{p}_\infty)\|_{V^*}^q-$$

$$-\frac{1}{q\varepsilon^q}\|[B(\omega,u_\infty,\mathbf{p}_\infty)](t)-B_\infty(\omega^*,u_\infty,\mathbf{p}_\infty)\|_{V^*}^q \tag{25}$$

with $\varepsilon>0$. On the right hand side of (24) by Young's inequality we obtain

$$|\langle[G(\omega)](t)-G_\infty(\omega^*),u(t)-u_\infty\rangle+\langle[F(\omega,u)](t),\mathbf{p}(t)-\mathbf{p}_\infty\rangle|\le$$

$$\le\frac{\varepsilon^p}{p}\|u(t)-u_\infty\|_V^p+\frac{\varepsilon^p}{p}\|\mathbf{p}(t)-\mathbf{p}_\infty\|_V^p+ \tag{26}$$

$$+\frac{1}{q\varepsilon^q}\|[G(\omega)](t)-G_\infty(\omega^*)\|_{V^*}^q+\frac{1}{q\varepsilon^q}\|[F(\omega,u)](t)\|_{V^*}^q$$

where the last two terms (as we have verified earlier) tend to 0 as $t\to\infty$. We show that last two terms of the right hand side on (25) converges to 0 as $t\to\infty$. Clearly,

$$\|[A(\omega,u_\infty,\mathbf{p}_\infty)](t)-A_\infty(\omega^*,u_\infty,\mathbf{p}_\infty)\|_{V^*}^q\le$$

$$\le\sum_{i=0}^n\int_\Omega|a_i(t,\cdot,\omega(t,\cdot),u_\infty,Du_\infty,\mathbf{p}_\infty,D\mathbf{p}_\infty)-$$

$$-a_{i,\infty}(\omega^*,u_\infty,Du_\infty,\mathbf{p}_\infty,D\mathbf{p}_\infty)|^q.$$

The integrand on the right hand side is a.e. convergent in Ω as $t\to\infty$ by condition A5 and since $\omega(t,x)\to\omega^*(x)$ for a.a. $x\in\Omega$. Further, by conditions A2, A5

$$|a_i(t,\cdot,\omega(t,\cdot),u_\infty,Du_\infty,\mathbf{p}_\infty,D\mathbf{p}_\infty)-a_{i,\infty}(\omega^*,u_\infty,Du_\infty,\mathbf{p}_\infty,D\mathbf{p}_\infty)|^q\le$$

$$\le\text{const}\cdot\left(\|c_1(\omega)\|_{L^\infty(Q_\infty)}+\|c_1(\omega^*)\|_{L^\infty(Q_\infty)}\right)\cdot$$

$$\cdot\left(|u_\infty|^p+|Du_\infty|^p+|\mathbf{p}_\infty|^p+|D\mathbf{p}_\infty|^p+\|k_1\|_{L^q(\Omega)}\right)$$

where the right hand side is integrable in $L^1(\Omega)$, so by Lebesgue's theorem we obtain $\|[A(\omega,u_\infty,\mathbf{p}_\infty)](t)-A_\infty(\omega,u_\infty,\mathbf{p}_\infty)\|_{V^*}^q\to 0$ as $t\to\infty$. The convergence of the last term in (25) can be proved similarly.

Now, by choosing sufficiently small ε in (24) and by using (25), (26) and the above convergences we obtain

$$y'(t)+\text{const}\cdot\|u(t)-u_\infty\|_V^p+\text{const}\cdot\|\mathbf{p}(t)-\mathbf{p}_\infty\|_V^p\le\varphi(t) \tag{27}$$

where $\varphi(t)\to 0$ as $t\to\infty$ and the constants are positive. By the continuous embedding $L^p(\Omega)\hookrightarrow L^2(\Omega)$

$$y'(t)+\text{const}\cdot y(t)^{\frac{p}{2}}+\text{const}\cdot\|\mathbf{p}(t)-\mathbf{p}_\infty\|_V^p\le\varphi(t).$$

It is not difficult to show that this inequality implies $\lim_{t\to\infty} y(t) = 0$ (see the proof of Theorem 2 in [13]).

By integrating (27) over $(T-1, T+1)$ we obtain

$$y(T+1) - y(T-1) + \text{const} \cdot \int_{T-1}^{T+1} \|u(t) - u_\infty\|_V^p \, dt +$$
$$+\text{const} \int_{T-1}^{T+1} \|\mathbf{p}(t) - \mathbf{p}_\infty\|_V^p \, dt \le \int_{T-1}^{T+1} \varphi(t)\, dt.$$

Clearly the right hand side tends to 0, and by the convergence of $y(t)$ the first difference on the left hand side tends to 0, too, which yields the desired convergences. The proof of stabilization is complete. ∎

6. Examples

Now we give some examples of functions which fulfil conditions of the previous theorems. Consider the following functions

$$\begin{aligned} &a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta) = \\ &\quad = \mathcal{P}(t,x)P(\xi)Q(\eta_0,\eta)\zeta_i|\zeta|^{p_1-2} + \tilde{\mathcal{P}}(t,x)\tilde{P}(\xi)\tilde{Q}(\eta_0,\eta)\zeta_i|\zeta|^{r_1-1}, \\ &\quad \text{for } i = 1,\dots,n, \end{aligned} \tag{28}$$

$$\begin{aligned} &a_0(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta) = \\ &= \mathcal{P}(t,x)P(\xi)Q(\eta_0,\eta)\zeta_0|\zeta_0|^{p_1-2} + \tilde{\mathcal{P}}_0(t,x)\tilde{P}_0(\xi)\tilde{Q}(\eta_0,\eta)\zeta_0|\zeta_0|^{r_1-1}, \end{aligned} \tag{29}$$

$$\begin{aligned} &b_i(t,x,\xi,\zeta_0,\eta_0,\eta) = \\ &\quad = \mathcal{R}(t,x)R(\xi)S(\zeta_0)\eta_i|\eta|^{p_2-2} + \tilde{\mathcal{R}}(t,x)\tilde{R}(\xi)\tilde{S}(\zeta_0)\eta_i|\eta|^{r_2-1}, \\ &\quad \text{for } i = 1,\dots,n, \end{aligned} \tag{30}$$

$$\begin{aligned} &b_0(t,x,\xi,\zeta_0,\eta_0,\eta) = \\ &= \mathcal{R}(t,x)R(\xi)S(\zeta_0)\eta_i|\eta|^{p_2-2} + \tilde{\mathcal{R}}(t,x)\tilde{R}(\xi)\tilde{S}(\zeta_0)\eta_0|\eta_0|^{r_2-1}, \end{aligned} \tag{31}$$

where

E1 a) $\mathcal{P}, \tilde{\mathcal{P}}, \tilde{\mathcal{P}}_0 \in L^\infty(Q_\infty)$, $P, \tilde{P}, \tilde{P}_0 \in C(\mathbb{R})$, $Q, \tilde{Q} \in C(\mathbb{R}^{n+1})$. Further, there exists constants $\alpha, \beta > 0$ such that $\mathcal{P}(t,x)P(\xi)Q(\eta_0,\eta) \ge \alpha$,

$|Q(\eta_0,\eta)| \leq \beta$, and $\tilde{Q}(\eta_0,\eta) \leq \beta \cdot \left(|(\eta_0,\eta)|^{\frac{p_2(p_1-1-r_1)}{p_1}}+1\right)$,

$\tilde{\mathcal{P}}(t,x)\tilde{P}(\xi)\tilde{Q}(\eta_0,\eta) \geq 0$ for a.a. $(t,x) \in Q_\infty$ and every $(\xi,\eta_0,\eta) \in \in \mathbb{R} \times \mathbb{R}^{n+1}$ where $0 \leq r_1 < p_1 - 1$.

b) Functions $\mathcal{R}, \tilde{\mathcal{R}} \in L^\infty(Q_\infty)$, $R, \tilde{R}, S, \tilde{S} \in C(\mathbb{R})$. Further, there exists constants $\alpha,\beta > 0$ such that $\mathcal{R}(t,x)R(\xi)S(\zeta_0) \geq \alpha$, $|S(\zeta_0)| \leq \beta$, and $\tilde{S}(\zeta_0) \leq \beta \cdot \left(|\zeta_0|^{\frac{p_1(p_2-1-r_2)}{p_2}}+1\right)$, $\tilde{\mathcal{R}}(t,x)\tilde{R}(\xi)\tilde{S}(\zeta_0) \geq 0$ for a.a. $(t,x) \in Q_\infty$ and every $(\xi,\zeta_0) \in \mathbb{R} \times \mathbb{R}$ where $0 \leq r_2 < p_2 - 1$.

Assuming E1/a, E1/b one can easily verify that functions (28)–(30) fulfil conditions A1–A4, B1–B4 (see, e.g., [4]). Further, if we make a minor modification in condition E1/b, namely let $\tilde{S}(\zeta_0) \leq \beta \left(|\zeta_0|^{\frac{2(p_2-1-r_2)}{p_2}}+1\right)$ for every $\zeta_0 \in \mathbb{R}^{n+1}$, then the above functions satisfy conditions A1–A4, B1–B3, B4*.

Now consider the following functions for $i = 0,\dots,n$

$$a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta) = \mathcal{P}(t,x)P(\xi)\zeta_i|(\zeta_0,\zeta,\eta_0,\eta)|^{p-2}, \tag{32}$$

$$b_i(t,x,\xi,\zeta_0,\eta_0,\eta) = \mathcal{R}(t,x)R(\xi)\eta_i|(\zeta_0,\eta_0,\eta)|^{p-2} \tag{33}$$

where

E2. Functions $\mathcal{P},\mathcal{R} \in L^\infty(Q_\infty)$, $P,R \in C(\mathbb{R})$ and there exists a positive constant α such that $\mathcal{P}(t,x)P(\xi) \geq \alpha$, $\mathcal{R}(t,x)R(\xi) \geq \alpha$ for a.a. $(t,x) \in \in Q_\infty$ and every $\xi \in \mathbb{R}$. Further, there exist functions $\mathcal{P}_*,\mathcal{R}_* \in L^\infty(\Omega)$ such that $\lim_{t\to\infty} \mathcal{P}(t,x) = \mathcal{P}_*(x)$ and $\lim_{t\to\infty} \mathcal{R}(t,x) = \mathcal{R}_*(x)$.

THEOREM 5. *Suppose* $2 \leq p \leq 4$ *and E2, then the above* (32)–(33) *functions satisfy conditions A1–A5, B1–B3, B4*, B5, AB with* $p_1 = p_2 = p$.

PROOF. It is easy to see that all the conditions instead of AB are satisfied. For the proof of AB see [5]. ∎

If we consider functions

$$a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta) = \zeta_i|(\zeta_0,\zeta)|^{p-2} + \mathcal{P}(t,x)P(\xi)\zeta_i|(\zeta_0,\zeta,\eta_0,\eta)|^{r-2},$$

$$b_i(t,x,\xi,\zeta_0,\eta_0,\eta) = \eta_i|(\eta_0,\eta)|^{p-2} + \mathcal{R}(t,x)R(\xi)\eta_i|(\zeta_0,\eta_0,\eta)|^{r-2}$$

where $1 \leq r \leq 4$ and E2 hold then it is easy to see that these functions satisfy conditions A1–A5, B1–B3, B4*, B5, AB with $p_1 = p_2 = p \geq \max\{2,r\}$.

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