Acta Math. Hungar., **127 (1–2)** (2010), 178–194.
DOI: 10.1007/s10474-009-9113-y
First published online December 18, 2009

ON A NONLINEAR SYSTEM CONSISTING OF THREE DIFFERENT TYPES OF DIFFERENTIAL EQUATIONS

Á. BESENYEI*

Department of Applied Analysis, Eötvös Loránd University, H-1117 Budapest, Pázmány P. s. 1/C, Hungary
e-mail: badam@cs.elte.hu

(Received May 15, 2009; revised July 31, 2009; accepted August 24, 2009)

Abstract. We consider a system consisting of a first order differential equation, a parabolic and an elliptic equation. Existence of weak solutions is proved by using the Schauder fixed point theorem. The paper improves some results of [3, 6] which is illustrated by examples.

1. Introduction

In this paper we consider a system consisting of three different types of differential equations: a first order ordinary differential equation, a parabolic and an elliptic equation. Such a system may occur, e.g, as a model of fluid flow in porous medium, i.e., a medium with lots of tiny holes, for example, limestone. The flow of a fluid through the medium is influenced by the large surface of the solid matrix and the closeness of the holes. If the fluid carries dissolved chemical species, chemical reactions can occur that can change the porosity. In [10] this process was modelled by J. Logan, M. R. Petersen and T. S. Shores by the following system of equations in one dimension:

$$\omega D_t u = D_x\big(\alpha|v|D_x u\big) + K(\omega)D_x p \cdot D_x u - kug(\omega) \tag{1.1}$$

$$D_t\omega = bug(\omega) \tag{1.2}$$

$$D_x\big(K(\omega)D_x p\big) = bug(\omega), \tag{1.3}$$

$$v = -K(\omega)D_x p \tag{1.4}$$

---

*Supported by grant OTKA T 049819 from the Hungarian National Foundation for Scientific Research.

Key words and phrases: flow in porous medium, system of partial differential equations, monotone operators, Schauder fixed point theorem.

2000 Mathematics Subject Classification: 35K60, 35J60.

in $\mathbb{R}^+ \times (0,1)$ with initial and boundary conditions

$$u(0,x) = u_0(x), \quad \omega(0,x) = \omega_0(x) \quad x \in (0,1),$$

$$u(t,0) = u_1(t), \quad D_x u(t,1) = 0 \quad t > 0,$$

$$p(t,0) = 1, \quad p(t,1) = 0 \quad t > 0$$

where ω is the porosity (i.e. the proportion of the holes), u is the concentration of the dissolved chemical solute carried by the fluid, p is the pressure, v is the velocity, further, α, k, b are given constants, K and g are given real functions. Clearly, equation (1.4) might be eliminated by substituting it into (1.1). Further, for fixed u, (1.2) is a first order ordinary differential equation with respect to ω; for fixed ω and p, (1.1) is a parabolic equation with respect to u; and for fixed ω and u, (1.3) is an elliptic equation with respect to p. So the above system is a hybrid evolutionary/elliptic problem. In [7] a similar model was considered by using the method of Rothe, further, some numerical experiments were also done without giving correct proof on the existence of solutions. In [6] the following generalization of the above model was investigated where the equations may contain nonlocal dependence on the unknowns:

$$\begin{aligned}(1.5)\qquad & D_t u(t,x) \\ & - \sum_{i=1}^n D_i\big[a_i\big(t,x,\omega(t,x),u(t,x),Du(t,x),\mathsf{p}(t,x),D\mathsf{p}(t,x);\omega,u,\mathsf{p}\big)\big] \\ & + a_0\big(t,x,\omega(t,x),u(t,x),Du(t,x),\mathsf{p}(t,x),D\mathsf{p}(t,x);\omega,u,\mathsf{p}\big) \\ & = g(t,x), \qquad u(0,x) = 0,\end{aligned}$$

$$(1.6)\qquad D_t\omega(t,x) = f\big(t,x,\omega(t,x),u(t,x);u\big), \qquad \omega(0,x) = \omega_0(x),$$

$$\begin{aligned}(1.7)\qquad & \sum_{i=1}^n D_i\big[b_i\big(t,x,\omega(t,x),u(t,x),\mathsf{p}(t,x),D\mathsf{p}(t,x);\omega,u,\mathsf{p}\big)\big] \\ & + b_0\big(t,x,\omega(t,x),u(t,x),\mathsf{p}(t,x),D\mathsf{p}(t,x);\omega,u,\mathsf{p}\big) = h(t,x),\end{aligned}$$

with some boundary conditions (which might be assumed to be homogeneous) for $(t,x) \in (0,T) \times \Omega$ where the terms after ";" represent the "nonlocal" variables (the variable p is written in sans-serif style in order to distinguish it and the sequence (p_k) from exponents p_1, p_2). Such nonlocal (or functional) dependence is motivated by diffusion models for heat or population, where the diffusion coefficient may depend on the unknowns in a nonlocal way, e.g., on the integral of certain quantities. These types of quasilinear parabolic equations were investigated in [8]. General types of parabolic quasilinear functional differential equations were considered in [11] by using the theory of monotone operators. The results of that paper were extended to systems of parabolic functional differential equations in [2] and to nonlocal parabolic variational inequalities in [5]. For details on applications of monotone type operators to parabolic and functional parabolic partial differential equations we refer to [12].

The present paper is continuation of the works [3, 4, 6] where a system similar to (1.5)–(1.7) was considered. The aim of this paper is to provide a different approach to the results of [3, 6], further, by modifying some assumptions of [3, 6] the results will apply to a larger class of nonlocal systems. This will be illustrated with some examples at the end of the paper.

Our main tool will be the Schauder fixed point theorem (see, e.g., [13]):

THEOREM 1.1. *Let X be a Banach space and $H \subset X$ be a bounded, closed, convex set. Suppose that $T : H \to H$ is a continuous and compact operator. Then there exists a fixed point of T, i.e., there is $x \in H$ such that $T(x) = x$.*

In Section 2 we shall make some assumptions on the functions a_i, b_i, f. Then we shall define the weak form of system (1.5)–(1.7) and prove existence of weak solutions in Section 3.

2. Formulation of the problem

Notation. Throughout the paper $\Omega \subset \mathbb{R}^n$ is a bounded domain with the cone property (see [1]), further, $0 < T < \infty$, $2 \leqq p_1, p_2 < \infty$ are real numbers. We briefly write $Q_T := (0,T) \times \Omega$. Let V_i be a closed linear subspace of the usual Sobolev space $W^{1,p_i}(\Omega)$ $(i = 1,2)$ which contains $W_0^{1,p_i}(\Omega)$ (the closure of $C_0^\infty(\Omega)$). Denote by $X_i := L^{p_i}(0,T;V_i)$ the space of measurable functions $v : (0,T) \to V_i$ for which $\|v\|_{V_i}^{p_i}$ is integrable and the norm of X_i is given by

$$\|v\|_{X_i} := \left(\int_0^T \left\| v(t) \right\|_{V_i}^{p_i} dt \right)^{1/p_i}.$$

It is well-known (see [13]) that $X_i^* = L^{q_i}(0,T;V_i)$ where $\frac{1}{p_i} + \frac{1}{q_i} = 1$. The pairing between V_i^* and V_i, further, between X_i^* and X_i will be denoted by $\langle \cdot, \cdot \rangle$ and $[\cdot, \cdot]$, respectively. Finally, $C(M,N)$ stands for the set of continuous functions $M \to N$.

Assumptions. We shall seek weak solutions u and p in X_1, X_2, respectively. Further, according to its original physical meaning, the space for ω

will be $L^\infty(Q_T)$. In what follows, ξ, (ζ_0,ζ), (η_0,η) refer to the variables ω, (u,Du), $(\mathsf{p},D\mathsf{p})$, respectively, further, w, v_1, v_2 to the nonlocal variables (the subindices referring to the corresponding spaces X_1, X_2).

(A1) The functions $a_i : Q_T\times\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}\times L^2(Q_T)\times L^{p_1}(Q_T)\times X_2\to\mathbb{R}$ $(i=0,\dots,n)$ are of Carathéodory type for fixed $(w,v_1,v_2)\in L^2(Q_T)\times L^{p_1}(Q_T)\times X_2$, i.e., they are measurable in $(t,x)\in Q_T$ and continuous in $(\xi,\zeta_0,\zeta,\eta_0,\eta)\in\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$.

(A2) There exist bounded operators $\delta_1 : L^2(Q_T)\times L^{p_1}(Q_T)\times X_2\to\mathbb{R}^+$ and $k_1 : L^2(Q_T)\times L^{p_1}(Q_T)\times X_2\to L^{q_1}(Q_T)$ such that k_1 is continuous and

$$\big|a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2)\big|$$
$$\leqq \delta_1(w,v_1,v_2)\Big(|\zeta_0|^{p_1-1}+|\zeta|^{p_1-1}+|\eta_0|^{\frac{p_2}{q_1}}+|\eta|^{\frac{p_2}{q_1}}+\big[k_1(w,v_1,v_2)\big](t,x)\Big),$$

for a.e. $(t,x)\in Q_T$ and all $(\xi,\zeta_0,\zeta,\eta_0,\eta)\in\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$, $(w,v_1,v_2)\in L^2(Q_T)\times L^{p_1}(Q_T)\times X_2$ $(i=0,\dots,n)$.

(A3) There exists a constant $C>0$ such that for a.e. $(t,x)\in Q_T$ and all $(\xi,\zeta_0,\zeta,\eta_0,\eta)$, $(\xi,\zeta_0,\tilde{\zeta},\eta_0,\eta)\in\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$, $(w,v_1,v_2)\in L^2(Q_T)\times L^{p_1}(Q_T)\times X_2$,

$$\sum_{i=0}^{n}\big(a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2)-a_i(t,x,\xi,\tilde{\zeta}_0,\tilde{\zeta},\eta_0,\eta;w,v_1,v_2)\big)(\zeta_i-\tilde{\zeta}_i)$$
$$\geqq C\cdot\big(|\zeta_0-\tilde{\zeta}_0|^{p_1}+|\zeta-\tilde{\zeta}|^{p_1}\big).$$

(A4) There exist a constant $c_2>0$ and a bounded operator $k_2 : L^{p_1}(Q_T)\to L^1(Q_T)$ such that

$$\sum_{i=0}^{n}a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2)\zeta_i\geqq c_2\big(|\zeta_0|^{p_1}+|\zeta|^{p_1}\big)-\big[k_2(v_1)\big](t,x)$$

for a.e. $(t,x)\in Q_T$ and all $(\xi,\zeta_0,\zeta,\eta_0,\eta)\in\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$, $(w,v_1,v_2)\in L^2(Q_T)\times L^{p_1}(Q_T)\times X_2$. Further,

$$\lim_{r\to+\infty}\sup_{\|v_1\|_{L^{p_1}(Q_T)}\leqq r}\frac{\big\|k_2(v_1)\big\|_{L^1(Q_T)}}{r^{p_1}}=0. \tag{2.1}$$

(A5) If $\omega_k\to\omega$ in $L^2(Q_T)$, $u_k\to u$ in $L^{p_1}(Q_T)$, further, $\mathsf{p}_k\to\mathsf{p}$ in X_2 then for all $(\zeta_0,\zeta)\in\mathbb{R}^{n+1}$ and a.e. $(t,x)\in Q_T$,

$$a_i(t,x,\omega_k,\zeta_0,\zeta,\mathsf{p}_k,D\mathsf{p}_k;\omega_k,u_k,\mathsf{p}_k)\to a_i(t,x,\omega,\zeta_0,\zeta,\mathsf{p},D\mathsf{p};\omega,u,\mathsf{p})$$

for a suitable subsequence.

(B1) The functions $b_i : Q_T \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1} \times L^\infty(Q_T) \times X_1 \to \mathbb{R}$ $(i = 0, \dots, n)$ are of Carathéodory type for fixed $(w, v_1) \in L^\infty(Q_T) \times X_1$, i.e., they are measurable in $(t, x) \in Q_T$ and continuous in $(\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$.

(B2) There exist a function $\hat{c}_1 \in C(\mathbb{R}, \mathbb{R}^+)$ and bounded operators $\hat{\delta}_1 : L^\infty(Q_T) \times X_1 \to \mathbb{R}^+$, $\hat{k}_1 : L^\infty(Q_T) \times X_1 \to L^{q_2}(Q_T)$ such that $\hat{k}_1$ is continuous and

$$\left| b_i(t, x, \xi, \zeta_0, \eta_0, \eta; w, v_1) \right|$$

$$\leqq \hat{\delta}_1(w, v_1)\hat{c}_1(\xi)\Big(|\eta_0|^{p_2-1} + |\eta|^{p_2-1} + |\zeta_0|^{\frac{p_1}{q_2}} + \big[\hat{k}_1(w, v_1)\big](t, x)\Big)$$

for a.e. $(t, x) \in Q_T$ and every $(\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$, $(w, v_1) \in L^\infty(Q_T) \times X_1$ $(i = 0, \dots, n)$.

(B3) There exists a constant $\hat{C} > 0$ such that for a.e. $(t, x) \in Q_T$ and all $(\xi, \zeta_0, \eta_0, \eta)$, $(\xi, \zeta_0, \tilde{\eta}_0, \tilde{\eta}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$, $(w, v_1) \in L^\infty(Q_T) \times X_1$

$$\sum_{i=0}^{n} \big(b_i(t, x, \xi, \zeta_0, \eta_0, \eta; w, v_1) - b_i(t, x, \xi, \zeta_0, \tilde{\eta}_0, \tilde{\eta}; w, v_1)\big)(\eta_i - \tilde{\eta}_i)$$

$$\geqq \hat{C} \cdot \big(|\eta_0 - \tilde{\eta}_0|^{p_2} + |\eta - \tilde{\eta}|^{p_2}\big).$$

(B4) There exist constants $\hat{c}_2 > 0$, $0 \leqq p_1' < p_2$, functions $\hat{\gamma} \in C(\mathbb{R}, \mathbb{R})$, $\hat{k}_2 \in L^1(Q_T)$ and a bounded operator $\hat{\Gamma} : L^\infty(Q_T) \to L^\infty(Q_T)$ such that

$$\sum_{i=0}^{n} b_i(t, x, \xi, \zeta_0, \eta_0, \eta; w, v_1)\eta_i$$

$$\geqq \hat{c}_2\big(|\eta_0|^{p_2} + |\eta|^{p_2}\big) - \hat{\gamma}(\xi)\big[\hat{\Gamma}(w)\big](t, x)\big(|\zeta_0|^{p_1'} + \hat{k}_2(t, x)\big)$$

for a.e. $(t, x) \in Q_T$ and all $(\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}, (w, v_1) \in L^\infty(Q_T) \times X_1$.

(B5) If (ω_k) is bounded in $L^\infty(Q_T)$, $\omega_k \to \omega$ a.e. in Q_T and $u_k \to u$ in $L^{p_1}(Q_T)$, then for all $(\eta_0, \eta) \in \mathbb{R}^{n+1}$ and a.e. $(t, x) \in Q_T$,

$$b_i(t, x, \omega_k, u_k, \eta_0, \eta; \omega_k, u_k) \to b_i(t, x, \omega, u, \eta_0, \eta; \omega, u)$$

for a suitable subsequence.

(F1) The function $f : Q_T \times \mathbb{R}^2 \times L^{p_1}(Q_T) \to \mathbb{R}$ is of Carathéodory type for fixed $v_1 \in L^{p_1}(Q_T)$, i.e., it is measurable in $(t, x) \in Q_T$ and continuous in $(\xi,\zeta_0)\in\mathbb{R}^2$. In addition, there exist a bounded operator $\mathcal{K}: L^{p_1}(Q_T)\to\mathbb{R}^+$ and a function $K\in C(\mathbb{R},\mathbb{R}^+)$ satisfying $\big|K(\zeta_0)\big|\leqq d\big(|\zeta_0|^{\varrho_1}+1\big)$ for all $\zeta_0\in\mathbb{R}$ with some constants $0\leqq\varrho_1<p_1$ and $d>0$ for which the following hold: for every finite interval $I\subset\mathbb{R}$ there exists a constant $L_I>0$ such that for a.e. $(t,x)\in Q_T$ and all $(\xi,\zeta_0),(\tilde{\xi},\zeta_0)\in I\times\mathbb{R}$, $v_1\in L^{p_1}(Q_T)$,

$$\big|f(t,x,\xi,\zeta_0;v_1)-f(t,x,\tilde{\xi},\zeta_0;v_1)\big|\leqq\mathcal{K}(v_1)K(\zeta_0)L_I\cdot|\xi-\tilde{\xi}|.$$

(F2) There exists $\omega^*\in L^\infty(\Omega)$ such that for a.e. $(t,x)\in Q_T$ and all $(\xi,\zeta_0)\in\mathbb{R}^2$, $v_1\in L^{p_1}(Q_T)$,

$$\big(\xi-\omega^*(x)\big)\cdot f(t,x,\xi,\zeta_0;v_1)\leqq 0.$$

(F3) If $u_k\to u$ in $L^{p_1}(Q_T)$ then for all $\xi\in\mathbb{R}$ and for a.e. $(t,x)\in Q_T$,

$$f(t,x,\xi,u_k;u_k)\to f(t,x,\xi,u;u)\quad\text{for a subsequence.}$$

REMARK 2.1. We emphasize that the following changes have been made compared to the assumptions of [3, 6]. The domain of functions a_i in variable w and v_1 is $L^2(Q_T)$ and $L^{p_1}(Q_T)$ instead of $L^\infty(Q_T)$ and X_1, respectively. Further, in (A3) we posed uniform monotonicity condition and in (A4) the coercivity condition (2.1) is a little stronger than that of [6]. Besides these, the nonlocal dependence on v_2 of the functions b_i is omitted which will guarantee some uniqueness properties for the individual equations of the system. Finally, concerning f, in (F1) the estimate on K is weakened, further, the Lipschitz condition in variable ζ_0 is omitted, and the "continuity" assumption (F3) is weaker than that of [6].

Weak formulation. If the above assumptions are satisfied, we may define operators $A: L^\infty(Q_T)\times X_1\times X_2\to X_1^*$, $B: L^\infty(Q_T)\times X_1\times X_2\to X_2^*$ for $v_i\in X_i$ $(i=1,2)$ as

$$\big[A(\omega,u,\mathsf{p}),v_1\big]:=\int_{Q_T}\sum_{i=1}^n a_i\big(t,x,\omega(t,x),u(t,x),$$

$$Du(t,x),\mathsf{p}(t,x),D\mathsf{p}(t,x);\omega,u,\mathsf{p}\big)\,D_iv_1(t,x)\,dt\,dx$$

$$+\int_{Q_T}a_0\big(t,x,\omega(t,x),u(t,x),Du(t,x),\mathsf{p}(t,x),D\mathsf{p}(t,x);\omega,u,\mathsf{p}\big)\,v_1(t,x)\,dt\,dx,$$

$$\big[B(\omega,u,\mathsf{p}),v_2\big]$$

$$:=\int_{Q_T}\sum_{i=1}^n b_i\big(t,x,\omega(t,x),u(t,x),\mathsf{p}(t,x),D\mathsf{p}(t,x);\omega,u\big)\,D_iv_2(t,x)\,dt\,dx$$

$$+\int_{Q_T} b_0\big(t,x,\omega(t,x),u(t,x),\mathsf{p}(t,x),D\mathsf{p}(t,x);\omega,u\big)\,v_2(t,x)\,dt\,dx.$$

In addition, let us introduce the operator of differentiation $L : D(L) \to X_1^*$ by

$$D(L) = \big\{u \in X_1 :\ D_t u \in X_1^*,\ u(0) = 0\big\}, \qquad Lu = D_t u.$$

Finally, supposing $g \in L^{q_1}(Q_T)$ and $h \in L^{q_2}(Q_T)$ we define $G \in X_1^*$, $H \in X_2^*$ by

$$[G, v_1] := \int_{Q_T} g(t,x)v_1(t,x)\,dt\,dx, \quad [H, v_2] := \int_{Q_T} h(t,x)v_2(t,x)\,dt\,dx.$$

(In fact one may consider arbitrary $G \in X_1^*, H \in X_2^*$ not necessarily having the above special form.) Then, by the above introduced operators, the weak form of system (1.5)–(1.7) is defined as

$$Lu + A(\omega, u, \mathsf{p}) = G \tag{2.2}$$

$$\omega(t,x) = \omega_0(x) + \int_0^t f\big(s,x,\omega(s,x),u(s,x);u\big)\,ds \ \text{ a.e. in } Q_T \tag{2.3}$$

$$B(\omega, u, \mathsf{p}) = H. \tag{2.4}$$

3. Existence of solutions

Our main result is

THEOREM 3.1. *Suppose that conditions* (A1)–(A5), (B1)–(B5), (F1)–(F3) *are fulfilled. Then for every $\omega_0 \in L^\infty(\Omega)$, $G \in X_1^*$, $H \in X_2^*$ there exist solutions $\omega \in L^\infty(Q_T)$, $u \in D(L)$, $\mathsf{p} \in L^{p_2}(0,T;V_2)$ of problem* (2.2)–(2.4).

Before the proof we make some preparations. The main idea of the proof is to employ Theorem 1.1 for a modification of the system (2.2)–(2.4). We first reformulate some statements regarding the solvability of these equations. In case of (2.3) we just repeat the same assertions (on uniqueness and continuous dependence of solutions) as in [6].

PROPOSITION 3.2. *Assume that conditions* (F1), (F2) *are satisfied. Then for every fixed $u \in X_1$ and $\omega_0 \in L^\infty(\Omega)$ there exists a unique solution $\omega \in L^\infty(Q_T)$ of the integral equation* (2.3). *Further, the following estimate holds: $\|\omega\|_{L^\infty(Q_T)} \leqq \|\omega_0\|_{L^\infty(\Omega)} + \|\omega^*\|_{L^\infty(\Omega)}$.*

PROOF. See the proof of [6, Proposition 2]. $\square$

PROPOSITION 3.3. *Assume* (F1)–(F3) *and let* $(u_k) \subset X_1$*. Further let* ω_k $(k \in \mathbb{N})$ *be the solution of* (2.3) *corresponding to* u_k*. If* $u_k \to u$ *in* $L^{p_1}(Q_T)$ *then* (ω_k) *is bounded in* $L^\infty(Q_T)$ *and* $\omega_k \to \omega$ *a.e. in* Q_T *for a suitable subsequence where* ω *is the solution of* (2.3) *corresponding to* u*. In particular,* $\omega_k \to \omega$ *in* $L^\alpha(Q_T)$ *for arbitrary* $1 \leqq \alpha < \infty$*, also for the original sequence.*

PROOF. We first note that by Proposition 3.2, (ω_k) is bounded in $L^\infty(Q_T)$. Since $u_k \to u$ in $L^{p_1}(Q_T)$, there exists a subsequence (for simplicity denoted the same way as the original) such that for a.e. $x \in \Omega$,

$$u_k(\cdot, x) \to u(\cdot, x) \quad \text{in} \quad L^{p_1}(0,T). \tag{3.1}$$

In addition, by condition (F3), for a.e. $(s,x) \in Q_T$,

$$f\big(s,x,\omega(s,x),u_k(s,x);u_k\big) \to f\big(s,x,\omega(s,x),u(s,x);u\big). \tag{3.2}$$

Now fix $x \in \Omega$ such that (3.2) holds for a.e. $s \in (0,T)$, further, (3.1) holds, as well. We shall show that $\omega_k(t,x) \to \omega(t,x)$ as $k \to \infty$, and then by the subsequence trick (see [13]) we are done. Clearly,

$$\begin{aligned} &\big|\omega_k(t,x) - \omega(t,x)\big| \qquad (3.3)\\ &\leqq \int_0^t \Big|f\big(s,x,\omega_k(s,x),u_k(s,x);u_k\big) - f\big(s,\omega(s,x),u_k(s,x);u_k\big)\Big|\,ds \\ &+ \int_0^t \Big|f\big(s,x,\omega(s,x),u_k(s,x);u_k\big) - f\big(s,x,\omega(s,x),u(s,x);u\big)\Big|\,ds \end{aligned}$$

By the boundedness of (ω_k) in $L^\infty(Q_T)$ one has a finite interval $I \subset \mathbb{R}$ for which $\omega_k(s,x) \in I$ for a.e. $s \in (0,t)$ and for all k. Then by (F1) there is a constant $L_I > 0$ such that

$$\begin{aligned} &\Big|f\big(s,x,\omega_k(s,x),u_k(s,x);u_k\big) - f\big(s,\omega(s,x),u_k(s,x);u_k\big)\Big| \\ &\leqq \big|\mathcal{K}(u_k)\big| \cdot \Big|K\big(u_k(s,x)\big)\Big| \cdot L_I \cdot \big|\omega_k(s,x) - \omega(s,x)\big| \\ &\leqq \text{const} \cdot \Big(\big|u_k(s,x)\big|^{\varrho_1} + 1\Big) \cdot \big|\omega_k(s,x) - \omega(s,x)\big| \end{aligned}$$

Thus Hölder's inequality with conjugate exponents $\frac{p_1}{\varrho_1}$ and $\alpha = \frac{p_1}{p_1 - \varrho_1}$ yields

$$\int_0^t \Big|f\big(s,x,\omega_k(s,x),u_k(s,x);u_k\big) - f\big(s,\omega(s,x),u_k(s,x);u_k\big)\Big|\,ds \tag{3.4}$$

$$\leqq \text{const} \cdot \left(\int_0^t \left| \omega_k(s,x) - \omega(s,x) \right|^{\alpha} ds \right)^{1/\alpha} \cdot \left(\left\| u_k(\cdot,x) \right\|_{L^{p_1}(0,T)}^{\varrho_1/p_1} + 1 \right)$$

$$\leqq \text{const} \cdot \left(\int_0^t \left| \omega_k(s,x) - \omega(s,x) \right|^{\alpha} ds \right)^{1/\alpha}$$

where in the last estimate we used the boundedness of $\big(u_k(\cdot,x)\big)$ in $L^{p_1}(0,T)$. Now we turn to the second term on the right-hand side of (3.3). We claim that

$$\int_0^T \left| f\big(s,x,\omega(s,x),u_k(s,x);u_k\big) - f\big(s,x,\omega(s,x),u(s,x);u\big) \right| ds \to 0. \tag{3.5}$$

First, due to (3.2), the integrand tends to 0 for a.e. $s \in (0,T)$. Further, the integrand is equi-integrable in $L^1(0,T)$. Indeed, condition (F1), the boundednesses of (ω_k) and (u_k) in $L^\infty(Q_T)$ and $L^{p_1}(Q_T)$, respectively, imply

$$\left| f\big(s,x,\omega(s,x),u_k(s,x);u_k\big) \right| \tag{3.6}$$

$$= \left| f\big(s,x,\omega(s,x),u_k(s,x);u_k\big) - f\big(s,x,\omega^*(x),u_k(s,x);u_k\big) \right|$$

$$\leqq \left| \mathcal{K}(u_k) \right| \cdot \left| K\big(u_k(s,x)\big) \right| \cdot L_I \cdot \left| \omega(s,x) - \omega^*(x) \right|$$

$$\leqq \text{const} \cdot \left(\left| u_k(s,x) \right|^{\varrho_1} + 1 \right) \leqq \text{const} \cdot \left(\left| u_k(s,x) \right|^{p_1} + 1 \right)$$

(note that $f\big(s,x,\omega^*(x),\zeta_0;v_1\big) = 0$ for all $\zeta_0 \in \mathbb{R}$ and $v_1 \in L^{p_1}(Q_T)$ due to condition (F3) and the continuity of f in ξ). By (3.1), the right-hand side of (3.6) is equi-integrable in $(0,T)$, thus the left-hand side is also equi-integrable. So Vitali's theorem yields (3.5). Now recalling (3.4) and (3.5), from (3.3) we obtain

$$\left| \omega_k(t,x) - \omega(t,x) \right|^{\alpha} \leqq \text{const} \cdot \left(\int_0^t \left| \omega_k(s,x) - \omega(s,x) \right|^{\alpha} ds + r_k(x) \right)$$

where $r_k(x) \to 0$ as $k \to \infty$. Therefore, by Gronwall's lemma we conclude that $\left| \omega_k(t,x) - \omega(t,x) \right|^{\alpha} \to 0$. $\square$

Now instead of (2.2) we consider a modified equation. Define the operator $\tilde{A}\colon X_1 \times L^2(Q_T) \times L^{p_1}(Q_T) \times X_2 \to X_1^*$ by

$$\big[\tilde{A}(\tilde{u},\omega,u,\mathsf{p}),v_1\big] := \int_{Q_T} \sum_{i=1}^n a_i\big(t,x,\omega(t,x),\tilde{u}(t,x),$$

$$D\tilde{u}(t,x),\mathsf{p}(t,x),D\mathsf{p}(t,x);\omega,u,\mathsf{p}\big)\, D_i v_1(t,x)\, dt\, dx$$

$$+\int_{Q_T} a_0\big(t,x,\omega(t,x),\tilde u(t,x),D\tilde u(t,x),\mathsf{p}(t,x),D\mathsf{p}(t,x);\omega,u,\mathsf{p}\big)\,v_1(t,x)\,dt\,dx$$

where $v_1 \in X_1$.

We rephrase and add uniqueness to [6, Proposition 4.].

Proposition 3.4. *Assume that conditions* (A1)–(A4) *are satisfied. Then for every fixed* $\omega \in L^2(Q_T)$, $u \in L^{p_1}(Q_T)$, $\mathsf{p} \in X_2$ *and* $G \in X_1^*$ *there exists a unique solution* $\tilde u \in D(L)$ *of problem* $L\tilde u + \tilde A(\tilde u,\omega,u,\mathsf{p}) = G$.

Proof. The assertion follows from the classical results, see [9, 13]. For fixed $\omega \in L^2(Q_T)$, $u \in L^{p_1}(Q_T)$ and $\mathsf{p} \in X_2$, conditions (A1)–(A4) imply that the operator $\tilde A(\cdot,\omega,u,\mathsf{p}) : X_1 \to X_1^*$ is bounded, demicontinuous, coercive and uniformly monotone which yield existence and uniqueness of solutions. $\square$

In case of (2.4) we also add uniqueness to [6, Proposition 5.].

Proposition 3.5. *Suppose that* (B1)–(B4) *hold. Then for every fixed* $\omega \in L^\infty(Q_T)$, $u \in X_1$ *and* $H \in X_2^*$ *there exists a unique solution* $\mathsf{p} \in X_2$ *of problem* $B(\omega,u,\mathsf{p}) = H$.

Proof. Similarly to the case of $\tilde A$, for fixed $\omega \in L^\infty(Q_T)$ and $u \in X_1$, the operator $B(\omega,u,\cdot) : X_2 \to X_2^*$ is bounded, demicontinuous, coercive and uniformly monotone. $\square$

Besides uniqueness, the uniform monotonicity implies also continuous dependence of solutions.

Proposition 3.6. *Assume conditions* (A1)–(A5). *Let* $\omega_k \to \omega$ *in* $L^2(Q_T)$, $u_k \to u$ *in* $L^{p_1}(Q_T)$, $\mathsf{p}_k \to \mathsf{p}$ *in* X_2 *and denote by* $\tilde u_k \in D(L)$ $(k \in \mathbb{N})$ *the unique solution of* $L\tilde u_k + \tilde A(\tilde u_k,\omega_k,u_k,\mathsf{p}_k) = G$ *where* $G \in X_1^*$. *Then* $\tilde u_k \to \tilde u$ *in* X_1 *where* $\tilde u$ *is the unique solution of* $L\tilde u + \tilde A(\tilde u,\omega,u,\mathsf{p}) = G$.

Proof. From condition (A3) we may deduce

$$C\|\tilde u_k - \tilde u\|_{X_1}^{p_1} \leqq \big[\tilde A(\tilde u_k,\omega_k,u_k,\mathsf{p}_k) - \tilde A(\tilde u,\omega_k,u_k,\mathsf{p}_k),\tilde u_k - \tilde u\big]$$
$$= \big[\tilde A(\tilde u,\omega,u,\mathsf{p}) - \tilde A(\tilde u,\omega_k,u_k,\mathsf{p}_k),\tilde u_k - \tilde u\big] - [L\tilde u_k - L\tilde u,\tilde u_k - \tilde u]. \tag{3.7}$$

Due to the monotonicity of the operator L, $[L\tilde u_k - L\tilde u,\tilde u_k - \tilde u] \geqq 0$. We show that $\big\|\tilde A(\tilde u,\omega,u,\mathsf{p}) - \tilde A(\tilde u,\omega_k,u_k,\mathsf{p}_k)\big\|_{X_1^*} \to 0$, then from inequality (3.7) it follows that $\|\tilde u_k - \tilde u\|_{X_1} \to 0$ (since $p_1 \geqq 2$). Now observe

$$\big\|\tilde A(\tilde u,\omega,u,\mathsf{p}) - \tilde A(\tilde u,\omega_k,u_k,\mathsf{p}_k)\big\|_{X_1^*}$$
$$\leqq \sum_{i=0}^n \big\|a_i(\cdot,\omega,\tilde u,D\tilde u,\mathsf{p},D\mathsf{p};\omega,u,\mathsf{p})$$
$$- a_i(\cdot,\omega_k,\tilde u,D\tilde u,\mathsf{p}_k,D\mathsf{p}_k;\omega_k,u_k,\mathsf{p}_k)\big\|_{L^{q_1}(Q_T)}. \tag{3.8}$$

By condition (A2), one has the estimate

$$\bigl| a_i(t,x,\omega_k,\tilde u, D\tilde u, \mathsf{p}_k, D\mathsf{p}_k;\omega_k,u_k,\mathsf{p}_k)\bigr|^{q_1}$$

$$\leqq \text{const}\cdot \delta_1(\omega_k,u_k,\mathsf{p}_k)\Bigl(|\tilde u|^{p_1}+|D\tilde u|^{p_1}+|\mathsf{p}_k|^{p_2}+|D\mathsf{p}_k|^{p_2}$$

$$+\bigl|k_1(\omega_k,u_k,\mathsf{p}_k)\bigr|^{q_1}\Bigr).$$

Note that the right-hand side of the above inequality is equi-integrable in $L^1(Q_T)$ by the strong convergence of (ω_k), (u_k), (p_k) and the boundedness of δ_1, k_1. Therefore, Vitali's theorem implies by (A5) that the right-hand side of (3.8) tends to 0 as $k\to\infty$ for a subsequence thus by subsequence trick (see [13]) it holds for the original sequence as well. $\square$

One may verify analogously

PROPOSITION 3.7. *Assume* (B1)–(B5). *Let* (ω_k) *be bounded in* $L^\infty(Q_T)$, *further,* $\omega_k\to\omega$ *a.e. in* Q_T, $u_k\to u$ *in* X_1 *and denote by* p_k $(k\in\mathbb{N})$ *the unique solution of* $B(\omega_k,u_k,\mathsf{p}_k)=H$ *where* $H\in X_2^*$. *Then* $\mathsf{p}_k\to\mathsf{p}$ *in* X_2 *where* $\mathsf{p}\in X_2$ *is the unique solution of* $B(\omega,u,\mathsf{p})=H$.

Now we turn to the proof of the main theorem.

PROOF OF THEOREM 3.1. For fixed $\omega\in L^2(Q_T)$, $u\in L^{p_1}(Q_T)$, $\mathsf{p}\in X_2$ we consider the following system:

$$L\tilde u+\tilde A(\tilde u,\omega,u,\mathsf{p})=G \tag{3.9}$$

$$\tilde\omega(t,x)=\omega_0(x)+\int_0^t f\bigl(s,x,\tilde\omega(s,x),\tilde u(s,x);\tilde u\bigr)\,ds \ \text{ a.e. in } Q_T \tag{3.10}$$

$$B(\tilde\omega,\tilde u,\tilde{\mathsf{p}})=H. \tag{3.11}$$

By Proposition 3.4, there exists a unique solution $\tilde u\in D(L)$ of (3.9). Now by substituting this $\tilde u$ into (3.10), Proposition 3.2 yields a unique solution $\omega\in L^\infty(Q_T)$ of that equation. Finally, with these $\tilde\omega$ and $\tilde u$ one obtains a unique $\tilde{\mathsf{p}}\in X_2$ that solves (3.11). So for every fixed $(\omega,u,\mathsf{p})\in L^2(Q_T)\times L^{p_1}(Q_T)\times X_2$, there exists a unique solution $(\tilde\omega,\tilde u,\tilde{\mathsf{p}})\in L^\infty(Q_T)\times D(L)\times X_2$ of system (3.9)–(3.11). Therefore, we may uniquely define an operator $\Phi: L^2(Q_T)\times L^{p_1}(Q_T)\times X_2\to L^2(Q_T)\times L^{p_1}(Q_T)\times X_2$ by $\Phi(\omega,u,\mathsf{p})=(\tilde\omega,\tilde u,\tilde{\mathsf{p}})$. We claim that the operator Φ is continuous, compact and there is a (bounded, closed, convex) ball $B(0,r)$ with radius r large enough such that $\Phi\bigl(B(0,r)\bigr)\subset B(0,r)$. Then by Theorem 1.1, Φ has a fixed point (in $B(0,r)$), i.e. $(\tilde\omega,\tilde u,\tilde{\mathsf{p}})=(\omega,u,\mathsf{p})$. Clearly, a fixed point of Φ is a solution to system (2.2)–(2.4) and also $\omega\in L^\infty(Q_T)$, $u\in D(L)$, $\mathsf{p}\in X_2$ hold since Φ in fact maps into $L^\infty(Q_T)\times D(L)\times X_2$.

1. *Continuity of* Φ. Let $\omega_k \to \omega$ in $L^2(Q_T)$, $u_k \to u$ in $L^{p_1}(Q_T)$, $\mathsf{p}_k \to \mathsf{p}$ in X_2 and denote $(\tilde\omega_k, \tilde u_k, \tilde{\mathsf{p}}_k) := \Phi(\omega_k, u_k, \mathsf{p}_k)$, $(\tilde\omega, \tilde u, \tilde{\mathsf{p}}) := \Phi(\omega, u, \mathsf{p})$. Then by Proposition 3.6, $\tilde u_k \to \tilde u$ in X_1 which implies by Proposition 3.3 that $\tilde\omega_k \to \tilde\omega$ in $L^2(Q_T)$, moreover $(\tilde\omega_k)$ is bounded in $L^\infty(Q_T)$. Finally, the convergence of $(\tilde\omega_k)$ and $(\tilde u_k)$ yield $\tilde{\mathsf{p}}_k \to \tilde{\mathsf{p}}$ in X_2 due to Proposition 3.7.

2. *Compactness of* Φ. Let $(\omega_k) \subset L^2(Q_T)$, $(u_k) \subset L^{p_1}(Q_T)$, $(\mathsf{p}_k) \subset X_2$ be bounded sequences and denote $(\tilde\omega_k, \tilde u_k, \tilde{\mathsf{p}}_k) := \Phi(\omega_k, u_k, \mathsf{p}_k)$. It suffices to show that there exist subsequences $(\tilde\omega_{k_l}) \subset (\tilde\omega_k)$, $(\tilde u_{k_l}) \subset (\tilde u_k)$, $(\tilde{\mathsf{p}}_{k_l}) \subset (\tilde{\mathsf{p}}_k)$ which are convergent in $L^2(Q_T)$, $L^{p_1}(Q_T)$ and X_2, respectively. To this end, we first show the boundedness of $(\tilde u_k)$ in X_1 and the boundedness of $(L\tilde u_k)$ in X_1^*.

By employing condition (A4) and the monotonicity of the operator L one obtains

$$
\begin{aligned}
[G, \tilde u_k] &= [L\tilde u_k, \tilde u_k] + \left[\tilde A(\tilde u_k, \omega_k, u_k, \mathsf{p}_k), \tilde u_k\right] \\
&\geqq \int_{Q_T} \left(c_2|\tilde u_k|^{p_1} + c_2|D\tilde u_k|^{p_1} - k_2(u_k)\right) \\
&\geqq \|\tilde u_k\|_{X_1}\left(c_2\|\tilde u_k\|_{X_1}^{p_1-1} - \frac{\left\| k_2(u_k)\right\|_{L^1(Q_T)}}{\|\tilde u_k\|_{X_1}}\right),
\end{aligned}
$$

whence

$$
\|\tilde u_k\|_{X_1}^{p_1-1}\left(1 - \frac{1}{c_2}\cdot\frac{\left\| k_2(u_k)\right\|_{L^1(Q_T)}}{\|\tilde u_k\|_{X_1}^{p_1}}\right) \leqq \frac{1}{c_2}\|G\|_{X_1^*}. \tag{3.12}
$$

Since (u_k) is a bounded sequence in $L^{p_1}(Q_T)$ and $k_2 : L^{p_1}(Q_T) \to L^1(Q_T)$ is a bounded operator thus $\left\| k_2(u_k)\right\|_{L^1(Q_T)} \leqq c$ for all k with some constant $c > 0$. So if there was a subsequence $(\tilde u_{k_l}) \subset (\tilde u_k)$ such that $\|\tilde u_{k_l}\|_{X_1} \to \infty$ as $l \to \infty$ then (3.12) would imply $\|\tilde u_{k_l}\|_{X_1}^{p_1-1} \leqq \frac{2}{c_2}\|G\|_{X_1^*}$ for all l large enough which contradicts the unboundedness of $(\tilde u_{k_l})$.

In order to show the boundedness of the sequence $(L\tilde u_k)$ in X_1^* we use Hölder's inequality and deduce

$$
\left\| [\tilde A(\tilde u_k, \omega_k, u_k, \mathsf{p}_k)\right\|_{X_2^*} \leqq \sum_{i=0}^{n} \left\| a_i(\cdot, \omega_k, \tilde u_k, D\tilde u_k, \mathsf{p}_k, D\mathsf{p}_k; \omega_k, u_k, \mathsf{p}_k)\right\|_{L^{q_1}(Q_T)}.
$$

Recalling condition (A2) it follows for all i that

$$
\left\| a_i(\cdot, \omega_k, \tilde u_k, D\tilde u_k, \mathsf{p}_k, D\mathsf{p}_k; \omega_k, u_k, \mathsf{p}_k)\right\|_{L^{q_1}(Q_T)}
$$

$$\leqq \text{const} \cdot \delta_1(\omega_k, u_k, \mathsf{p}_k)\Big(\|\tilde{u}_k\|_{X_1}^{\frac{p_1}{q_1}} + \|\mathsf{p}_k\|_{X_2}^{\frac{p_2}{q_1}} + \big\|k_1(\omega_k, u_k, \mathsf{p}_k)\big\|_{L^{q_1}(Q_T)}\Big).$$

Due to the boundedness of the sequences $(\tilde{u}_k)$, (ω_k), (u_k), (p_k) and the boundedness of the operators c_1, δ_1, k_2, the right-hand side of the above inequality is bounded. Hence we conclude that $\|L\tilde{u}_k\|_{X_1^*} = \big\|\tilde{A}(\tilde{u}_k, \omega_k, u_k, \mathsf{p}_k) + G\big\|_{X_1^*}$ $\leqq$ const, so $(L\tilde{u}_k)$ is a bounded sequence in X_1^*.

Now the well known compact embedding theorem (see [13]) implies that there exist a subsequence $(\tilde{u}_{k_l}) \subset (\tilde{u}_k)$ and a function $\tilde{u} \in X_1$ such that $\tilde{u}_{k_l} \to \tilde{u}$ weakly in X_1 and (strongly) in $L^{p_1}(Q_T)$. Then from Proposition 3.3 it follows that $\tilde{\omega}_{k_l} \to \tilde{\omega}$ in $L^2(Q_T)$ where $\tilde{\omega} \in L^\infty(Q_T)$ is the unique solution of

$$\tilde{\omega}(t,x) = \omega_0(x) + \int_0^t f\big(s, x, \tilde{\omega}(s,x), \tilde{u}(s,x); \tilde{u}\big)\, ds.$$

Finally, by the convergence of $(\tilde{\omega}_{k_l})$ and $(\tilde{u}_{k_l})$, from Proposition 3.7 it follows $\tilde{\mathsf{p}}_{k_l} \to \tilde{\mathsf{p}}$ where $\tilde{\mathsf{p}} \in X_2$ is the unique solution of $B(\tilde{\omega}, \tilde{u}, \tilde{\mathsf{p}}) = H$. So we obtained the convergent subsequences $(\tilde{\omega}_{k_l})$, $(\tilde{u}_{k_l})$ and $(\tilde{\mathsf{p}}_{k_l})$, as it was required.

3. $\Phi\big(B(0,r)\big) \subset B(0,r)$ *for all* $r > 0$ *large enough.* We shall show that for all $r > 0$ sufficiently large, if $\|\omega\|_{L^2(Q_T)} \leqq r$, $\|u\|_{L^{p_1}(Q_T)} \leqq r$ and $\|\mathsf{p}\|_{X_2} \leqq r$ then $\|\tilde{\omega}\|_{L^2(Q_T)} \leqq \frac{r}{3}$, $\|\tilde{u}\|_{L^{p_1}(Q_T)} \leqq \frac{r}{3}$ and $\|\tilde{\mathsf{p}}\|_{X_2} \leqq \frac{r}{3}$ where $(\tilde{\omega}, \tilde{u}, \tilde{\mathsf{p}}) = \Phi(\omega, u, \mathsf{p})$. Therefore, the (bounded, closed, convex) ball $B(0,r) \subset L^2(Q_T) \times L^{p_1}(Q_T) \times X_2$ is mapped into itself by Φ.

First, due to Proposition 3.2, $\|\tilde{\omega}\|_{L^\infty(Q_T)} \leqq \|\omega_0\|_{L^\infty(\Omega)} + \|\omega^*\|_{L^\infty(\Omega)}$ whence $\|\tilde{\omega}\|_{L^2(Q_T)} \leqq \lambda(\Omega)^{\frac{1}{2}} \|\tilde{\omega}\|_{L^\infty(Q_T)} \leqq \frac{r}{3}$ if

$$r \geqq 3\lambda(\Omega)^{\frac{1}{2}}\big(\|\omega_0\|_{L^\infty(\Omega)} + \|\omega^*\|_{L^\infty(\Omega)}\big)$$

(independently of $\|\omega\|_{L^\infty(Q_T)}$, $\|u\|_{L^{p_1}(Q_T)}$, $\|\mathsf{p}\|_{X_2}$).

In the case of $\tilde{u}$ we argue by contradiction. Suppose that there exist a sequence $r_k \to \infty$ and functions $\omega_k \in L^2(Q_T)$, $\mathsf{p}_k \in X_2$, $u_k \in L^{p_1}(Q_T)$ such that $\|\omega_k\|_{L^2(Q_T)} \leqq r_k$, $\|u_k\|_{L^{p_1}(Q_T)} \leqq r_k$, $\|\mathsf{p}_k\|_{X_2} \leqq r_k$ but $\|\tilde{u}_k\|_{L^{p_1}(Q_T)} > \frac{r_k}{3}$ (where $(\tilde{\omega}_k, \tilde{u}_k, \tilde{\mathsf{p}}_k) = \Phi(\omega_k, u_k, \mathsf{p}_k)$). By assumption (2.1),

$$\sup_{\|v_1\|_{L^{p_1}(Q_T)} \leqq r_k} \big\|k_2(v_1)\big\|_{L^1(Q_T)} \leqq \frac{c_2}{2 \cdot 3^{p_1}} \cdot {r_k}^{p_1}$$

for all k sufficiently large. Then by employing the obvious estimate

$$\|u_k\|_{L^{p_1}(Q_T)} \leqq \|u_k\|_{X_1}$$

in (3.12) it follows (by $\|\tilde{u}_k\|_{L^{p_1}(Q_T)} > \frac{r_k}{3}$) that

$$\text{(3.13)} \qquad \frac{1}{c_2}\|G\|_{X_1^*} \geqq \|\tilde{u}_k\|_{X_1}^{p_1-1}\left(1-\frac{1}{c_2}\cdot\frac{\|k_2(u_k)\|_{L^1(Q_T)}}{\|\tilde{u}_k\|_{X_1}^{p_1}}\right)$$

$$\geqq \|\tilde{u}_k\|_{L^{p_1}(Q_T)}^{p_1-1}\left(1-\frac{1}{c_2}\cdot\frac{\|k_2(u_k)\|_{L^1(Q_T)}}{\|\tilde{u}_k\|_{L^{p_1}(Q_T)}^{p_1}}\right) \geqq \frac{1}{2}\|\tilde{u}_k\|_{L^{p_1}(Q_T)}^{p_1-1}.$$

Hence $\|\tilde{u}_k\|_{L^{p_1}(Q_T)} \leqq \frac{r_k}{3}$ for all $r_k \geqq 3\left(\frac{2}{c_2}\|G\|_{X_1^*}\right)^{\frac{1}{p_1-1}}$ which contradicts the choice of $\tilde{u}_k$. So for all r sufficiently large, $\|u\|_{L^{p_1}(Q_T)} \leqq r$ implies $\|\tilde{u}\|_{L^{p_1}(Q_T)} \leqq \frac{r}{3}$ for all $u \in L^{p_1}(Q_T)$ (independently of $\|\omega\|_{L^\infty(Q_T)}$, $\|\mathsf{p}\|_{X_2}$).

Finally, in the case of $\tilde{\mathsf{p}}$ we may use similar arguments as above. Assume that there exist a sequence $r_k \to \infty$ and functions $\omega_k \in L^2(Q_T)$, $u_k \in L^{p_1}(Q_T)$, $\mathsf{p}_k \in X_2$ such that

$$\|\omega\|_{L^2(Q_T)} \leqq r_k, \quad \|u_k\|_{L^{p_1}(Q_T)} \leqq r_k, \quad \|\mathsf{p}_k\|_{X_2} \leqq r_k$$

but $\|\tilde{\mathsf{p}}_k\|_{X_2} > \frac{r_k}{3}$ (where $(\tilde{\omega}_k, \tilde{u}_k, \tilde{\mathsf{p}}_k) = \Phi(\omega, u, \mathsf{p})$). Note that $(\tilde{\omega}_k)$ is bounded in $L^\infty(Q_T)$, further, $\|\tilde{u}_k\| \leqq \frac{r_k}{3}$ holds for all k large enough (accordingly to the previous part of the proof). Then from condition (B4) (similarly to estimate (3.13)) we may deduce

$$\|\tilde{\mathsf{p}}_k\|_{X_2}^{p_2-1} \leqq \frac{1}{\hat{c}_2}\|H\|_{X_2^*} + \frac{\|\hat{\gamma}(\tilde{\omega}_k)\hat{\Gamma}(\tilde{\omega}_k)\|_{L^\infty(Q_T)}}{\hat{c}_2}\cdot\frac{\|\tilde{u}_k\|_{L^{p_1}(Q_T)}^{p_1'} + \|\hat{k}_2\|_{L^1(Q_T)}}{\|\tilde{\mathsf{p}}_k\|_{X_2}}$$

$$\leqq \frac{1}{\hat{c}_2}\left(\|H\|_{X_2^*} + \frac{3c\|\hat{k}_2\|_{L^1(Q_T)}}{r_k}\right) + \frac{c}{\hat{c}_2}\cdot\left(\frac{r_k}{3}\right)^{p_1'-1} \leqq \left(\frac{r_k}{3}\right)^{p_2-1}$$

if $\frac{1}{\hat{c}_2}\left(\|H\|_{X_2^*} + \frac{3c\|\hat{k}_2\|_{L^1(Q_T)}}{r_k}\right) \leqq \frac{1}{2}\left(\frac{r_k}{3}\right)^{p_2-1}$ and $\frac{c}{\hat{c}_2}\left(\frac{r_k}{3}\right)^{p_1'} \leqq \frac{1}{2}\left(\frac{r_k}{3}\right)^{p_2}$ which obviously hold for all k sufficiently large since $1 \leqq p_1' < p_2$. Thus $\|\tilde{\mathsf{p}}_k\|_{X_2} \leqq \frac{r_k}{3}$ for large k which is a contradiction. $\square$

4. Examples

We now show some examples for functions satisfying conditions (A1)–(A5), (B1)–(B5). Let the functions a_i, b_i have the form

$$\text{(4.1)} \qquad a_i(t, x, \xi, \zeta_0, \zeta, \eta_0, \eta; w, v_1, v_2)$$

$$= \big[\pi(w)\big](t,x)\big[\varphi(v_1)\big](t,x)\big[\psi(v_2)\big](t,x)P(\xi)Q(\eta_0,\eta)\zeta_i\big|(\zeta_0,\zeta)\big|^{p_1-2},$$

$$b_i(t,x,\xi,\zeta_0,\eta_0,\eta;w,v_1) \tag{4.2}$$

$$= \big[\kappa(w)\big](t,x)\big[\lambda(v_1)\big](t,x)R(\xi)S(\zeta_0)\eta_i\big|(\eta_0,\eta)\big|^{p_2-2}$$

where the following hold.

(E1) The operators $\pi : L^2(Q_T) \to L^\infty(Q_T)$, $\varphi : L^{p_1}(Q_T) \to L^\infty(Q_T)$ and $\psi : X_2 \to L^\infty(Q_T)$ are bounded, further, if $\omega_k \to \omega$ in $L^2(Q_T)$, $u_k \to u$ in $L^{p_1}(Q_T)$, $\mathsf{p}_k \to \mathsf{p}$ in X_2 then $\pi(\omega_k) \to \pi(\omega)$, $\varphi(u_k) \to \varphi(u)$, $\psi(\mathsf{p}_k) \to \psi(\mathsf{p})$ a.e. in Q_T for a suitable subsequence. In addition, $P \in C(\mathbb{R},\mathbb{R}^+) \cap L^\infty(\mathbb{R})$, $Q \in C(\mathbb{R}^{n+1},\mathbb{R}^+) \cap L^\infty(\mathbb{R}^{n+1})$ and there exists a positive lower bound for the values of π, φ, ψ, P, Q.

(E2) The operators $\kappa : L^\infty(Q_T) \to L^\infty(Q_T)$, $\lambda : L^{p_1}(Q_T) \to L^\infty(Q_T)$ are bounded. Further, if (ω_k) is bounded in $L^\infty(Q_T)$, $\omega_k \to \omega$ a.e. in Q_T, $u_k \to u$ in $L^{p_1}(Q_T)$ then $\kappa(\omega_k) \to \kappa(\omega)$, $\lambda(u_k) \to \lambda(u)$ a.e. in Q_T for a suitable subsequence. In addition, $R \in C(\mathbb{R},\mathbb{R}^+)$, $S \in C(\mathbb{R},\mathbb{R}^+) \cap L^\infty(\mathbb{R})$ and there exists a positive lower bound for the values of κ, λ, R, S.

PROPOSITION 4.1. *Assume that conditions* (E1)–(E2) *are satisfied. Then the functions* (4.1)–(4.2) *fulfil conditions* (A1)–(A5), (B1)–(B5).

By using Young's and Hölder's inequalities it is not difficult to prove the above statement, for similar arguments, see [2].

The operator π may have the form

$$\big[\pi(w)\big](t,x) = \Phi\bigg(\int_{Q_t}|w|^\beta\bigg), \tag{4.3}$$

(where $Q_t = (0,t) \times \Omega$) or

$$\big[\pi(w)\big](t,x) = \Phi\bigg(\int_0^t \big|w(s,x)\big|^\beta\,ds\bigg), \tag{4.4}$$

$$\big[\pi(w)\big](t,x) = \Phi\bigg(\int_\Omega \big|w(t,\xi)\big|^\beta\,d\xi\bigg), \tag{4.5}$$

where $1 \leqq \beta \leqq 2$, $\Phi \in C(\mathbb{R},\mathbb{R}^+) \cap L^\infty(\mathbb{R})$ and $\Phi \geqq \text{const} > 0$ (in case of (4.3) Φ does not need to be bounded from above). Further, the operators φ may have, e.g., one of the forms

$$\big[\varphi(v)\big](t,x) = \Phi\bigg(\int_H |v|^\beta\bigg) \quad \text{or} \quad \big[\varphi(v)\big](t,x) = \Phi\bigg(\int_H jv\bigg) \tag{4.6}$$

where $1 \leqq \beta \leqq p_1$, $j \in L^{q_1}(Q_T)$, $\Phi \in C(\mathbb{R}, \mathbb{R}^+) \cap L^\infty(\mathbb{R})$, $\Phi \geqq \text{const} > 0$, further, the set of integration H may be Q_t, $(0,t)$ or Ω. Similarly, ψ may have the form

$$\left[\psi(v)\right](t,x) = \Psi\left(\int_H |v|^\beta + |Dv|^\beta\right) \quad \text{or} \quad \left[\psi(v)\right](t,x) = \Psi\left(\int_H d_1 v + d_2|Dv|\right)$$

where $1 \leqq \beta \leqq p_2$, $d_1, d_2 \in L^{q_2}(Q_T)$, $\Psi \in C(\mathbb{R}, \mathbb{R}^+) \cap L^\infty(\mathbb{R})$ and $\Psi \geqq \text{const} > 0$, further, the set H might be as above. Note that ψ may depend on the derivative of its argument.

For the operator κ one may consider similar examples as (4.3), (4.4) with arbitrary $1 \leqq \beta$, further, λ may have a similar form as φ above.

It is not difficult to prove that the above operators fulfil conditions (E1)–(E2); one can proceed by using similar arguments as for the examples in [3, 6].

As an example for a function f consider, e.g.,

$$f(t,x,\xi,\zeta_0;v) = -\left[\varphi(v)\right](t,x)\tilde{f}(\zeta_0)\big(\xi - \omega^*(x)\big)$$

where $\varphi : L^{p_1}(Q_T) \to L^\infty(Q_T)$ is nonnegative, bounded and if $u_k \to u$ in $L^{p_1}(Q_T)$ then $\varphi(u_k) \to \varphi(u)$ a.e. in Q_T for a subsequence (φ may have the form (4.6)). In addition, $\tilde{f} : \mathbb{R} \to \mathbb{R}$ is nonnegative, continuous and $\left|\tilde{f}(\zeta_0)\right| \leqq \text{const} \cdot \big(|\zeta_0|^{\varrho_1} + 1\big)$ for some $0 \leqq \varrho_1 < p_1$.

Remark 4.2. Note that conditions (E1)–(E2) apply for a larger class of functional dependence than in [6]. Indeed, in the present case also the main parts of the operators A, B may contain functional dependence of type (4.4)–(4.5), i.e., an integral on $(0,t)$ or Ω. In [6] the main part may contain only (4.3) type of nonlocal dependence.

References

[1] R. A. Adams, *Sobolev Spaces,* Academic Press (New York – San Francisco – London, 1975).

[2] Á. Besenyei, On systems of parabolic functional differential equations, *Annales Univ. Sci. Budapest. Eötvös Sect. Math.,* **47** (2004), 143–160.

[3] Á. Besenyei, Existence of weak solutions of a nonlinear system modelling fluid flow in porous media, *Electron. J. Diff. Eqns.,* **2006** (2006), 1–19.

[4] Á. Besenyei, Stabilization of solutions to a nonlinear system modelling fluid flow in porous media, *Annales Univ. Sci. Budapest. Eötvös Sect. Math.,* **49** (2006), 115–136.

[5] Á. Besenyei, On nonlinear parabolic variational inequalities containing nonlocal terms, *Acta Math. Hungar.,* **116** (2007), 145–162.

[6] Á. Besenyei, On a nonlinear system containing nonlocal terms related to a fluid flow model, *E.J. Qualitative Theory of Diff. Equ., Proc. 8'th Coll. Qualitative Theory of Diff. Equ.,* **3** (2008), 1–13.

[7] S. Cinca, Diffusion und Transport in porösen Medien bei veränderlichen Porosität, Diplomawork, University of Heidelberg (2000).

[8] M. Chipot and L. Molinet, Asymptotic behaviour of some nonlocal diffusion problems, *Appl. Anal.,* **80** (2001), 279–315.

[9] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires,* Dunod, Gauthier-Villars (Paris, 1969).

[10] J. D. Logan, M. R. Petersen and T. S. Shores, Numerical study of reaction-mineralogy-porosity changes in porous media, *Appl. Math. Comput.,* **127** (2002), 149–164.

[11] L. Simon, On parabolic functional differential equations of general divergence form, in: *Proceedings of the Conference FSDONA 04* (Milovy, 2004), 280–291.

[12] L. Simon, Application of monotone type operators to parabolic and functional parabolic PDE's, in: C. M. Dafermos, M. Pokorný (eds.), *Handbook of Differential Equations: Evolutionary Equations,* vol 4., North-Holland (Amsterdam, 2008), pp. 267–321.

[13] E. Zeidler, *Nonlinear Functional Analysis and its Applications I–II,* Springer (1990).