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EXISTENCE OF SOLUTIONS OF A NONLINEAR SYSTEM MODELLING FLUID FLOW IN POROUS MEDIA

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Abstract. We investigate the existence of weak solutions for nonlinear differential equations that describe fluid flow through a porous medium. Existence is proved using the theory of monotone operators, and some examples are given.

1. Introduction

In this paper we study a system of nonlinear differential equations that describes the flow of a fluid through a porous medium. A porous medium, roughly speaking, is a solid medium with lots of tiny holes. For example think of limestone. Such medium consists of two parts, the solid matrix and the holes. The flow of a fluid through the medium is influenced by the relatively large surface of the solid matrix and the closeness of the holes. If the fluid carries dissolved chemical species, a variety of chemical reactions can occur. Among these include reactions that can change the porosity. This process was modelled by Logan, Petersen, Shores [8] by the following system of equations in one dimension:

$$\begin{aligned} \omega(t,x)u_t(t,x) = \alpha\cdot(|v(t,x)|u_x(t,x))_x \\ + K(\omega(t,x))p_x(t,x)u_x(t,x) - ku(t,x)g(\omega(t,x)) \end{aligned} \tag{1.1}$$

$$\omega_t(t,x) = bu(t,x)g(\omega(t,x)) \tag{1.2}$$

$$(K(\omega(t,x))p_x(t,x))_x = bu(t,x)g(\omega(t,x)), \tag{1.3}$$

$$v(t,x) = -K(\omega(t,x))p_x(t,x), \quad t>0,\ x\in(0,1), \tag{1.4}$$

with initial and boundary conditions

$$\begin{gathered} u(0,x)=u_0(x), \quad \omega(0,x)=\omega_0(x) \quad x\in(0,1), \\ u(t,0)=u_1(t), \quad u_x(t,1)=0 \quad t>0, \\ p(t,0)=1, \quad p(t,1)=0 \quad t>0 \end{gathered}$$

where ω is the porosity, u is the concentration of the dissolved solute, p is the pressure, v is the velocity, further, α, k, b are given constants, K and g are given

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real functions. For the details of making this model see the cited paper and the references there. Observe, that v is explicitly given by ω and p in equation (1.4) thus we can eliminate equation (1.4) by substituting it into (1.1). Further, observe that for fixed u equation (1.2) is an ordinary differential equation with respect to the function ω; for fixed ω and p equation (1.1) is a parabolic problem with respect to the function u; and for fixed ω and u equation (1.3) is an elliptic problem with respect to the function p. This shows that the above system is a hybrid evolutionary/elliptic problem, thus theorems of ,,usual" systems of partial differential equations do not work. In [5] a similar model was considered by using the method of Rothe, further, some numerical experiments were done, however correct proof on existence of solutions were not made (and one can hardly find papers dealing with such kind of systems in rigorous mathematical way). In the following, we investigate a generalization of the above system by using the theory of operators of monotone type. We define the weak form of the system and prove existence of weak solutions. The main idea consists of two parts. First the choice of the appropriate spaces for the weak solutions (for the elliptic equation it will be not the usual space because of the time dependence). The second is the idea of the proof which is to apply the so called successive approximation (known e.g. from the theory of ordinary differential equations) and combine this with methods of the theory of monotone operators (can be found, e.g., in [2, 3, 4, 6, 7, 9, 10]. At the end of this paper, some examples are given.

1.1. **Notation.** In this section we introduce some notation. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the uniform C^1 regularity property (see [1]), further, let $0 < T < \infty$, $2 \le p_1, p_2 < \infty$ be real numbers. In the following, $Q_T := (0,T) \times \Omega$. Denote by $W^{1,p_i}(\Omega)$ the usual Sobolev space with the norm

$$\|v\|_{W^{1,p_i}(\Omega)} = \Big(\int_\Omega (|v|^{p_i} + \sum_{j=1}^n |D_j v|^{p_i}) \Big)^{1/p_i}$$

where D_j denotes the distributional derivative with respect to the j-th variable (later we use the notation $D = (D_1, \dots, D_n)$). In addition, let V_i be a closed linear subspace of the space $W^{1,p_i}(\Omega)$ which contains $W_0^{1,p_i}(\Omega)$ (the closure of $C_0^\infty(\Omega)$ in $W^{1,p_i}(\Omega)$), and let $L^{p_i}(0,T;V_i)$ be the Banach space of measurable functions $u\colon (0,T) \to V_i$ such that $\|u\|_{V_i}^{p_i}$ is integrable and the norm is given by

$$\|u\|_{L^{p_i}(0,T;V_i)} = \Big(\int_0^T \|u(t)\|_{V_i}^{p_i} dt \Big)^{1/p_i}.$$

The dual space of $L^{p_i}(0,T;V_i)$ is $L^{q_i}(0,T;V_i^*)$ where $\frac{1}{p_i} + \frac{1}{q_i} = 1$ and V_i^* is the dual space of V_i. In what follows, we use the notation $X_i := L^{p_i}(0,T;V_i)$. The pairing between X_i^* and X_i is denoted by $[\cdot,\cdot]$, and $D_t u$ stands for the derivative (with respect to the variable t) of a function $u \in L^{p_i}(0,T;V_i)$. It is well known (see [10]) that if $D_t u \in X_i^*$ then $u \in C([0,T], L^2(\Omega))$ so that $u(0)$ makes sense.

1.2. **Statement of the problem.** Let us consider the following nonlinear system (in Q_T) which is the generalization of the system introduced in the first section:

$$D_t\omega(t,x) = f(t,x,\omega(t,x),u(t,x)), \quad \omega(0,x)=\omega_0(x), \tag{1.5}$$

$$\begin{aligned} &D_t u(t,x) - \sum_{i=1}^n D_i\left[a_i(t,x,\omega(t,x),u(t,x),Du(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x))\right] \\ &+ a_0(t,x,\omega(t,x),u(t,x),Du(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x)) \\ &= g(t,x,\omega(t,x)), \quad u(0,x)=0, \end{aligned} \tag{1.6}$$

$$\begin{aligned} &-\sum_{i=1}^n D_i[b_i(t,x,\omega(t,x),u(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x))] \\ &+ b_0(t,x,\omega(t,x),u(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x)) \\ &= h(t,x,\omega(t,x),u(t,x)) \end{aligned} \tag{1.7}$$

with boundary conditions homogenous Dirichlet or Neumann, for example

$$\sum_{i=1}^n a_i(t,x,\omega(t,x),u(t,x),Du(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x))\nu_i = 0,$$
$$\mathbf{p}(t,x)=0 \quad x\in\partial\Omega, t>0,$$

where ν is the unit normal along the boundary. (The variable $\mathbf{p}$ is written by boldface letter for the purpose of distinguishing it from exponents p_1, p_2). Moreover, if $\partial\Omega = S_1\cup S_2$ where $S_1\cap S_2=\emptyset$, then we can pose different boundary conditions on the elements of the partition. That is the case in the model (1.1)–(1.4) where the partitions are the endpoints of the interval $[0,1]$. Clearly, we can assume the boundary conditions to be homogeneous by subtracting a suitable function from the unknown function. The above system is indeed a generalization of the problem (1.1)–(1.4), since (as we showed in the introduction) v can be eliminated form (1.1)–(1.4), and in Proposition 2.5 we show that by some assumptions the solution ω of equation (1.2) is strictly positive hence we can divide equation (1.1) by ω. By using this observation that the above equations are three types of differential equations we can give natural conditions on functions a_i, b_i, f, g, h which (as we will see) imply existence of weak solutions of the above system. Before giving these assumptions let us introduce a notation. In the following, a vector $\xi\in\mathbb{R}^{n+1}$ is written in the form $\xi=(\zeta_0,\zeta)$ where $\zeta_0\in\mathbb{R}$ and $\zeta=(\zeta_1,\dots,\zeta_n)\in\mathbb{R}^n$.

Assumptions.

(A1) Functions $a_i\colon Q_T\times\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}\to\mathbb{R}$ $(i=0,\dots,n)$ are Carathéodory functions, i.e. they are measurable in $(t,x)\in Q_T$ for every $(\omega,u,Du,\mathbf{p},D\mathbf{p})$ in $\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ and continuous in $(\omega,u,Du,\mathbf{p},D\mathbf{p})\in\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ for a.a. $(t,x)\in Q_T$.

(A2) There exist a continuous function $c_1\colon\mathbb{R}\to\mathbb{R}^+$ and a function $k_1\in L^{q_1}(Q_T)$ such that

$$\begin{aligned} &|a_i(t,x,\omega,u,Du,\mathbf{p},D\mathbf{p})| \\ &\le c_1(\omega)\Big(|u|^{p_1-1}+|Du|^{p_1-1}+|\mathbf{p}|^{\frac{p_2}{q_1}}+|D\mathbf{p}|^{\frac{p_2}{q_1}}+k_1(t,x)\Big), \end{aligned}$$

for a.a. $(t,x)\in Q_T$ and every $(\omega,u,Du,\mathbf{p},D\mathbf{p})\in\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ $(i=0,\dots,n)$.

(A3) There exists a constant $C > 0$ such that for a.a. $(t,x) \in Q_T$ and every $(\omega, \zeta_0, \zeta, \mathbf{p}, D\mathbf{p}), (\omega, \zeta_0, \eta, \mathbf{p}, D\mathbf{p}) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$

$$\sum_{i=1}^{n} \left(a_i(t,x,\omega,\zeta_0,\zeta,\mathbf{p},D\mathbf{p}) - a_i(t,x,\omega,\zeta_0,\eta,\mathbf{p},D\mathbf{p})\right)(\zeta_i - \eta_i) \geq C \cdot |\zeta - \eta|^{p_1}.$$

(A4) There exist a constant $c_2 > 0$, a continuous function $\gamma \colon \mathbb{R} \to \mathbb{R}$ and a function $k_2 \in L^1(Q_T)$ such that

$$\sum_{i=0}^{n} a_i(t,x,\omega,\zeta_0,\zeta,\mathbf{p},D\mathbf{p})\zeta_i \geq c_2\left(|\zeta_0|^{p_1} + |\zeta|^{p_1}\right) - \gamma(\omega)k_2(t,x)$$

for a.a. $(t,x) \in Q_T$ and every $(\omega, \zeta_0, \zeta, \mathbf{p}, D\mathbf{p}) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$.

(B1) Functions $b_i \colon Q_T \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1} \to \mathbb{R}$ $(i = 0, \dots, n)$ are Carathéodory functions, i.e. they are measurable in $(t,x) \in Q_T$ for every $(\omega, u, \mathbf{p}, D\mathbf{p}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$ and continuous in $(\omega, u, \mathbf{p}, D\mathbf{p}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$ for a.a. $(t,x) \in Q_T$.

(B2) There exist a continuous function $\hat{c}_1 \colon \mathbb{R} \to \mathbb{R}^+$ and a function $\hat{k}_1 \in L^{q_2}(Q_T)$ such that

$$|b_i(t,x,\omega,u,\mathbf{p},D\mathbf{p})| \leq \hat{c}_1(\omega)\left(|\mathbf{p}|^{p_2-1} + |D\mathbf{p}|^{p_2-1} + |u|^{\frac{p_1}{q_2}} + \hat{k}_1(t,x)\right)$$

for a.a. $(t,x) \in Q_T$ and every $(\omega, u, \mathbf{p}, D\mathbf{p}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$ $(i = 0, \dots, n)$.

(B3) There exists a constant $\hat{C} > 0$ such that for a.a. $(t,x) \in Q_T$ and every $(\omega, u, \zeta_0, \zeta), (\omega, u, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$

$$\begin{aligned}&\sum_{i=0}^{n} \left(b_i(t,x,\omega,u,\zeta_0,\zeta) - b_i(t,x,\omega,u,\eta_0,\eta)\right)(\zeta_i - \eta_i)\\ &\geq \hat{C} \cdot \left(|\zeta_0 - \eta_0|^{p_2} + |\zeta - \eta|^{p_2}\right).\end{aligned}$$

(B4) There exist a constant $\hat{c}_2 > 0$, a continuous function $\hat{\gamma} \colon \mathbb{R} \to \mathbb{R}$ and a function $\hat{k}_2 \in L^1(Q_T)$ such that

$$\sum_{i=0}^{n} b_i(t,x,\omega,u,\zeta_0,\zeta)\zeta_i \geq \hat{c}_2\left(|\zeta_0|^{p_2} + |\zeta|^{p_2}\right) - \hat{\gamma}(\omega)\left(|u|^{p_1} + \hat{k}_2(t,x)\right)$$

for a.a. $(t,x) \in Q_T$ and every $(\omega, u, \zeta_0, \zeta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$.

(F1) Function $f \colon Q_T \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function, i.e. it is measurable in $(t,x) \in Q_T$ for every fixed $(\omega, u) \in \mathbb{R}^2$ and continuous in $(\omega, u) \in \mathbb{R}^2$ for a.a. $(t,x) \in Q_T$. Further, for every bounded set $I \subset \mathbb{R}$ there exists a continuous function $K_1 \colon \mathbb{R} \to \mathbb{R}^+$ such that
   (i) there exist nonnegative constants d_1, d_2 such that $|K_1(u)| \leq d_1|u|^{\frac{p_1}{q_2}} + d_2$ for all $u \in \mathbb{R}$,
   (ii) for a.a. $(t,x) \in Q_T$ and every $(\omega, u), (\tilde{\omega}, u) \in I \times \mathbb{R}$

$$|f(t,x,\omega,u) - f(t,x,\tilde{\omega},u)| \leq K_1(u) \cdot |\omega - \tilde{\omega}|.$$

(F2) There exists a continuous function $K_2 \colon \mathbb{R} \to \mathbb{R}^+$ such that for a.a. $(t,x) \in Q_T$ and every (ω, u), $(\omega, \tilde{u}) \in \mathbb{R}^2$

$$|f(t,x,\omega,u) - f(t,x,\omega,\tilde{u})| \leq K_2(\omega) \cdot |u - \tilde{u}|.$$

(F3) There exists $\omega^* \in L^\infty(\Omega)$ such that for a.a. $(t,x) \in Q_T$ and every $(\omega, u) \in \mathbb{R}^2$, $(\omega - \omega^*(x)) \cdot f(t,x,\omega,u) \leq 0$.

(G1) Function $g\colon Q_T \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, i.e. it is measurable in $(t,x) \in Q_T$ for every $\omega \in \mathbb{R}$ and continuous in $\omega \in \mathbb{R}$ for a.a. $(t,x) \in Q_T$.

(G2) There exist a continuous function $c_3\colon \mathbb{R} \to \mathbb{R}^+$ and a function $k_3 \in L^{q_1}(Q_T)$ such that
$$|g(t,x,\omega)| \le c_3(\omega) k_3(t,x)$$
for a.a. $(t,x) \in Q_T$ and every $\omega \in \mathbb{R}$.

(H1) Function $h\colon Q_T \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function, i.e. it is measurable in $(t,x) \in Q_T$ for every $(\omega,u) \in \mathbb{R}^2$ and continuous in $(\omega,u) \in \mathbb{R}^2$ for a.a. $(t,x) \in Q_T$.

(H2) There exist a continuous function $c_4\colon \mathbb{R} \to \mathbb{R}^+$ and a function $k_4 \in L^{q_2}(Q_T)$ such that
$$|h(t,x,\omega,u)| \le c_4(\omega)\left(|u|^{\frac{p_1}{q_2}} + k_4(t,x)\right)$$
for a.a. $(t,x) \in Q_T$ and every $(\omega,u) \in \mathbb{R}^2$.

2. Weak form

Let us define the operators $A\colon L^\infty(Q_T) \times X_1 \times X_2 \to X_1^*$, $B\colon L^\infty(Q_T) \times X_1 \times X_2 \to X_2^*$, $G\colon L^\infty(Q_T) \to X_1^*$, $H\colon L^\infty(Q_T) \times X_1 \to X_2^*$ as follows:
$$\begin{aligned}
&[A(\omega,u,p),v]\\
&:= \int_{Q_T} \sum_{i=1}^n a_i(t,x,\omega(t,x),u(t,x),Du(t,x)\mathbf{p}(t,x),D\mathbf{p}(t,x))D_i v(t,x)\,dt\,dx\\
&\quad + \int_{Q_T} a_0(t,x,\omega(t,x),u(t,x),Du(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x))v(t,x)\,dt\,dx,
\end{aligned}$$
$$\begin{aligned}
[B(\omega,u,\mathbf{p}),v] &:= \int_{Q_T} \sum_{i=1}^n b_i(t,x,\omega(t,x),u(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x))D_i v(t,x)\,dt\,dx\\
&\quad + \int_{Q_T} b_0(t,x,\omega(t,x),u(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x))v(t,x)\,dt\,dx,
\end{aligned}$$
$$[G(\omega),v] := \int_{Q_T} g(t,x,\omega(t,x))v(t,x)\,dt\,dx,$$
$$[H(\omega,u),v] := \int_{Q_T} h(t,x,\omega(t,x),u(t,x))v(t,x)\,dt\,dx.$$
In addition, let us introduce the linear operator $L : D(L) \to X_1^*$ by the formula
$$D(L) = \{u \in X_1\colon D_t u \in X_1^*,\ u(0) = 0\}, \quad Lu = D_t u.$$
With the operators introduced above, we define the weak form of system (1.5)–(1.7) as
$$\omega(t,x) = \omega_0(x) + \int_0^t f(s,x,\omega(s,x),u(s,x))\,ds \tag{2.1}$$
$$Lu + A(\omega,u,\mathbf{p}) = G(\omega) \tag{2.2}$$
$$B(\omega,u,\mathbf{p}) = H(\omega,u). \tag{2.3}$$
It is well known (see e.g. [7]) that one gets the above weak forms by considering sufficiently smooth solutions and then using Green's theorem, after that one considers the equations in the spaces X_i. It is clear that if the boundary condition is

homogenous Neumann than $V_i = W^{1,p_i}(\Omega)$ (since the boundary term vanishes in Green's theorem) and if we have homogeneous Dirichlet boundary condition then $V_i = W_0^{1,p_i}(\Omega)$ (in order to eliminate the boundary terms in Green's theorem). Further, if we have a partition then for example in our one dimensional equation (1.1) with homogenous boundary conditions $V_1 = \{v \in W^{1,p_1}(0,1) : v(t,0) = 0\}$, and in addition $V_2 = W_0^{1,p_2}(0,1)$. In the next section we prove that the earlier introduced assumptions imply existence of solutions of the above system.

2.1. **Existence of solutions.**

Theorem 2.1. *Suppose that conditions (A1)–(A4), (B1)–(B4), (F1)–(F3), (G1)–(G2), (H1)–(H2) are fulfilled. Then for every* $\omega_0 \in L^\infty(\Omega)$ *there exists a solution* $\omega \in L^\infty(Q_T)$, $u \in D(L)$, $\mathbf{p} \in L^{p_2}(0,T;V_2)$ *of problem* (2.1)–(2.3).

First we prove some statements which we will use in the proof of the theorem.

Proposition 2.2. *Assume that conditions (F1), (F3) are satisfied. Then for every fixed* $u \in L^{p_1}(Q_T)$ *and* $\omega_0 \in L^\infty(Q_T)$ *there exists a unique solution* $\omega \in L^\infty(Q_T)$ *of the integral equation* (2.1)*, further, for the solution the estimate* $\|\omega\|_{L^\infty(Q_T)} \le \|\omega_0\|_{L^\infty(\Omega)} + \|\omega^*\|_{L^\infty(\Omega)}$ *holds.*

Proof. First we make an observation which we will use many times in the paper. Namely, from (F3) and the continuity of f, it follows that $f(t,x,\omega^*(x),u) = 0$ for a.a. $(t,x) \in Q_T$ and every $u \in \mathbb{R}$. Now note that if equation (2.1) has got a solution ω then ω is continuous in variable t (moreover it is absolutely continuous). Now let us fix a point $x \in \Omega$. If $\omega(t_0,x) > \omega^*(x)$ for some $t_0 \in (0,T)$ then $\omega(t,x) > \omega^*(x)$ for all $t \in [t_0, t_0+\varepsilon]$ where ε is sufficiently small. Then by condition (F3) we obtain $f(t,x,\omega(t,x),u(t,x)) \le 0$ thus

$$\begin{aligned}\omega(t,x) &= \omega_0(x) + \int_0^t f(s,x,\omega(s,x),u(s,x))\,ds \\ &= \omega_0(x) + \int_0^{t_0} f(s,x,\omega(s,x),u(s,x))\,ds + \int_{t_0}^t f(s,x,\omega(s,x),u(s,x))\,ds \\ &\le \omega_0(x) + \int_0^{t_0} f(s,x,\omega(s,x),u(s,x))\,ds \\ &= \omega(t_0,x),\end{aligned}$$

that is, ω is decreasing in variable t. Similar to this, if $\omega(t_0,x) < \omega^*(x)$ for some $t_0 > 0$ then ω is locally increasing in t. From this it is easy to see that $\omega(t,x) \in [\omega^*(x), \omega_0(x)]$ (or $[\omega_0(x), \omega^*(x)]$) for a.a. $(t,x) \in Q_T$ thus $|\omega(t,x)| \le |\omega_0(x)| + |\omega^*(x)|$ for a.a. $(t,x) \in Q_T$ hence $\|\omega\|_{L^\infty(Q_T)} \le \|\omega_0\|_{L^\infty(\Omega)} + \|\omega^*\|_{L^\infty(\Omega)}$.

Let us define a function $\tilde{f}\colon Q_T \times \mathbb{R}^2 \to \mathbb{R}$ by

$$\begin{aligned}&\tilde{f}(t,x,\omega,u) \\ &= \begin{cases} f(t,x,\omega,u), & \text{if } |\omega| \le \|\omega_0\|_{L^\infty(\Omega)} + \|\omega^*\|_{L^\infty(\Omega)}, \\ f(t,x,\|\omega_0\|_{L^\infty(\Omega)} + \|\omega^*\|_{L^\infty(\Omega)},u), & \text{if } \omega \ge \|\omega_0\|_{L^\infty(\Omega)} + \|\omega^*\|_{L^\infty(\Omega)}, \\ f(t,x,-\|\omega_0\|_{L^\infty(\Omega)} - \|\omega^*\|_{L^\infty(\Omega)},u), & \text{if } \omega \le -\|\omega_0\|_{L^\infty(\Omega)} - \|\omega^*\|_{L^\infty(\Omega)}, \end{cases}\end{aligned}$$

and consider the following problem instead of (2.1):

$$\omega(t,x) = \omega_0(x) + \int_0^t \tilde{f}(s,x,\omega(s,x),u(s,x))\,ds. \tag{2.4}$$

Obviously $\tilde{f}$ also fulfils condition (F2), (F3), further, by choosing interval

$$I = \left[-\|\omega_0\|_{L^\infty(\Omega)} - \|\omega^*\|_{L^\infty(\Omega)}, \|\omega_0\|_{L^\infty(\Omega)} + \|\omega^*\|_{L^\infty(\Omega)}\right]$$

in condition (F1), we obtain that with some function K_1

$$|\tilde{f}(t,x,\omega,u) - \tilde{f}(t,x,\tilde{\omega},u)| \le K_1(u) \cdot |\omega - \tilde{\omega}|$$

for a.a. $(t,x) \in Q_T$ and every $(\omega,u) \in \mathbb{R}^2$, since f was extended as a constant function outside of I. This means that function $\tilde{F}$ satisfies condition (F1) globally. It is clear that if problem (2.4) has got a solution ω then $\|\omega\|_{L^\infty(Q_T)} \le \|\omega_0\|_{L^\infty(\Omega)} + \|\omega^*\|_{L^\infty(\Omega)}$. Since $\tilde{f}$ equals with f on interval I, every solution of (2.4) is a solution of (2.1) and converse. From the above arguments it follows that it is sufficient to show that the problem (2.4) has a unique solution $\omega \in L^\infty(Q_T)$. In other words, we may assume that condition (F1) is fulfilled by function f globally.

Existence. We use the method of successive approximation. Let $\omega_0(t,x) := \omega_0(x)$ ($(t,x) \in Q_T$), further,

$$\omega_{k+1}(t,x) := \omega_0(x) + \int_0^t f(s,x,\omega_k(s,x),u(s,x))\,ds. \tag{2.5}$$

Now fix a point $x \in \Omega$. We show that with suitable constant $c_x > 0$

$$|\omega_{k+1}(t,x) - \omega_k(t,x)| \le \left(\|\omega_0\|_{L^\infty(\Omega)} + \|\omega^*\|_{L^\infty(\Omega)}\right) \cdot c_x^{k+1} \frac{t^{\frac{k+1}{p_2}}}{[(k+1)!]^{1/p_2}}. \tag{2.6}$$

We proceed by induction on k. For $k=0$ we have

$$\begin{aligned}
|\omega_1(t,x) - \omega_0(t,x)|\,dt &= \Big|\int_0^t f(s,x,\omega_0(x),u(s,x))\,ds\Big| \\
&= \Big|\int_0^t \left(f(s,x,\omega_0(x),u(s,x)) - f(s,x,\omega^*(x),u(s,x))\right) ds\Big| \\
&\le \int_0^t |K_1(u(s,x))| \cdot |\omega_0(x) - \omega^*(x)|\,ds \\
&\le \left(\|\omega_0\|_{L^\infty(\Omega)} + \|\omega^*\|_{L^\infty(\Omega)}\right) \cdot \int_0^t |K_1(u(s,x))|\,ds.
\end{aligned}$$

Using condition (F1), Hölder's inequality and $u \in L^{p_1}(Q_T)$ we obtain

$$\begin{aligned}
\int_0^t |K_1(u(s,x))|\,ds &\le \Big(\int_0^T |K_1(u(s,x))|^{q_2} ds\Big)^{1/q_2} \cdot \Big(\int_0^t 1^{p_2}\Big)^{1/p_2} \\
&\le \Big(\int_0^t \Big(d_1|u(s,x)|^{\frac{p_1}{q_2}} + d_2\Big)^{q_2} ds\Big)^{1/q_2} \cdot t^{1/p_2} \\
&\le \text{const} \cdot \Big(\int_0^T (|u(s,x)|^{p_1} + 1)\,ds\Big)^{1/q_2} \cdot t^{1/p_2} \\
&= c_x \cdot t^{1/p_2}.
\end{aligned} \tag{2.7}$$

From the above two estimate follows (2.6) for $k=0$. Now let us suppose that estimate (2.6) holds for $k-1$. Then using condition (F1) and the assumption of induction we get

$$
\begin{aligned}
&|\omega_{k+1}(t,x)-\omega_k(t,x)| \\
&\le \int_0^t |f(s,x,\omega_k(s,x),u(s,x))-f(s,x,\omega_{k-1}(s,x),u(s,x))|ds \\
&\le \int_0^t |K_1(u(s,x))|\cdot|\omega_k(s,x)-\omega_{k-1}(s,x)|\,ds \\
&\le \int_0^t \big(|K_1(u(s,x))|\cdot\big(\|\omega_0\|_{L^\infty(\Omega)}+\|\omega^*\|_{L^\infty(\Omega)}\big)\cdot c_x^k\cdot\frac{s^{\frac{k}{p_2}}}{(k!)^{\frac{1}{p_2}}}\big)\,ds \\
&\le \big(\|\omega_0\|_{L^\infty(\Omega)}+\|\omega^*\|_{L^\infty(\Omega)}\big)\cdot c_x^k\cdot\Big(\int_0^T|K_1(u(s,x))|^{q_2}ds\Big)^{1/q_2}\cdot\Big(\int_0^t\frac{s^k}{k!}ds\Big)^{1/p_2} \\
&\le \big(\|\omega_0\|_{L^\infty(\Omega)}+\|\omega^*\|_{L^\infty(\Omega)}\big)\cdot c_x^{k+1}\cdot\frac{t^{\frac{k+1}{p_2}}}{[(k+1)!]^{1/p_2}}.
\end{aligned}
$$

The induction is complete. From estimate (2.6) it follows that for a.a. $x\in\Omega$ and every $t\in(0,T)$

$$
|\omega_{k+l}(t,x)-\omega_k(t,x)|\le\sum_{i=k+1}^{k+l}\big(\|\omega_0\|_{L^\infty(\Omega)}+\|\omega^*\|_{L^\infty(\Omega)}\big)\cdot c_x^i\frac{T^{\frac{i}{p_2}}}{(i!)^{1/p_2}}\to 0
$$

as $k,l\to\infty$, hence $(\omega_k(t,x))$ is a Cauchy sequence, therefore it is convergent to some $\omega(t,x)$, $\omega_k\to\omega$ a.e. in Q_T, moreover $\omega_k(\cdot,x)\to\omega(\cdot,x)$ in $L^\infty(0,T)$ for a.a. $x\in\Omega$. We show that ω is a solution of equation (2.1). Observe that the left hand side of the recurrence (2.5) converges to ω a.e. in Q_T, so it suffices to show that the right hand side of (2.5) a.e. tends to the right hand side of equation (2.1). But this is true since

$$
\begin{aligned}
&\Big|\int_0^t(f(s,x,\omega(s,x),u(s,x))-f(s,x,\omega_k(s,x),u(s,x)))ds\Big| \\
&\le\int_0^t|K_1(u(s,x))|\cdot|\omega(s,x)-\omega_k(s,x)|ds \\
&\le\int_0^T|K_1(u(s,x))|ds\cdot\|\omega(\cdot,x)-\omega_k(\cdot,x)\|_{L^\infty(0,T)} \\
&\le c_x\cdot\|\omega(\cdot,x)-\omega_k(\cdot,x)\|_{L^\infty(0,T)}\to 0\quad\text{as } k\to\infty.
\end{aligned}
$$

Uniqueness. Suppose that $\omega,\tilde\omega\in L^\infty(Q_T)$ are solutions of problem (2.1). Then by condition (F1)

$$
\begin{aligned}
|\omega(t,x)-\tilde\omega(t,x)|&\le\int_0^t|f(s,x,\omega(s,x),u(s,x))-f(s,x,\tilde\omega(s,x),u(s,x))|ds \\
&\le\int_0^t|K_1(u(s,x))|\cdot|\omega(s,x)-\tilde\omega(s,x)|\,ds \\
&\le\|K_1(u(\cdot,x))\|_{L^{q_2}(Q_T)}\cdot\Big(\int_0^t|\omega(s,x)-\tilde\omega(s,x)|^{p_2}\,ds\Big)^{1/p_2}
\end{aligned}
$$

hence

$$
|\omega(t,x)-\tilde\omega(t,x)|^{p_2}\le c_x^{p_2}\cdot\int_0^t|\omega(s,x)-\tilde\omega(s,x)|^{p_2}ds.
$$

Then Gronwall's lemma yields $|\omega(t,x) - \tilde{\omega}(t,x)| = 0$ for a.a. $(t,x) \in Q_T$ which means that $\omega = \tilde{\omega}$. □

Proposition 2.3. *Assume (F1)–(F3) and let $(u_k) \subset L^{p_1}(Q_T)$, further, let ω_k be the solution of* (2.1) *corresponding to u_k. If $u_k \to u$ in $L^{p_1}(Q_T)$ then $\omega_k \to \omega$ a.e. in Q_T where ω is the solution of* (2.1) *corresponding to u.*

Proof. Suppose that $u_k \to u$ in $L^{p_1}(Q_T)$. Then for a.a. $x \in \Omega$ $u_k(\cdot,x) \to u(\cdot,x)$ in $L^{p_1}(0,T)$. Let $\omega_k, \omega \in L^\infty(Q_T)$ be the corresponding solutions of (2.1). Now fix a point $x \in \Omega$. Since (ω_k) is bounded in $L^\infty(Q_T)$ by Proposition 2.2, we can apply condition (F1), (F2) and we obtain

$$
\begin{aligned}
&|\omega_k(t,x) - \omega(t,x)| \\
&\le \int_0^t |f(s,x,\omega_k(s,x),u_k(s,x)) - f(s,x,\omega(s,x),u(s,x))|\,ds \\
&\le \int_0^t [|K_1(u_k(s,x))| \cdot |\omega_k(s,x) - \omega(s,x)| + |K_2(\omega(s,x))| \cdot |u_k(s,x) - u(s,x)|]\,ds \\
&\le \Big(\int_0^t |K_1(u_k(s,x))|^{q_2} ds\Big)^{1/q_2} \cdot \Big(\int_0^t |\omega_k(s,x) - \omega(s,x)|^{p_2}\Big)^{1/p_2} ds \\
&\quad + \|K_2(\omega(\cdot,x))\|_{L^\infty(0,T)} \cdot \int_0^T |u_k(s,x) - u(s,x)|\,ds.
\end{aligned}
$$

By choosing $u = u_k$ and $t = T$ in estimate (2.7) and by using the convergence of $u_k(\cdot,x)$ in $L^{p_1}(0,T)$ we get that the first term containing u_k on the right hand side of the above inequality is bounded. In addition, by using the continuity of function K_2 it follows that $\|K_2(\omega(\cdot,x))\|_{L^\infty(0,T)}$ is finite. From the above arguments we obtain

$$
\begin{aligned}
&|\omega_k(t,x) - \omega(t,x)|^{p_2} \\
&\le \text{const} \cdot \int_0^t |\omega_k(s,x) - \omega(s,x)|^{p_2} ds + \text{const} \cdot \|u_k(\cdot,x) - u(\cdot,x)\|^{p_2}_{L^1(0,T)}.
\end{aligned}
$$

By Gronwall's lemma $|\omega_k(t,x) - \omega(t,x)|^{p_2} \le \text{const} \cdot \|u_k(\cdot,x) - u(\cdot,x)\|^{p_2}_{L^1(0,T)} \to 0$ as $k \to \infty$ which immediately yields the a.e. convergence of (ω_k). □

Remark 2.4. Since (ω_k) is bounded in $L^\infty(Q_T)$ and a.e. convergent, from Lebesgue's theorem it follows that (ω_k) is convergent in $L^\alpha(Q_T)$ for arbitrary $1 \le \alpha < \infty$.

Proposition 2.5. *Suppose that conditions (F1)–(F3) hold, further, $|w_0| > 0$ a.e. in Ω and $\omega_0 \cdot \omega^* \ge 0$ (that is they have the same sign). Then for the solution ω of* (2.1) *$|\omega(t,x)| > 0$ holds for a.a. $(t,x) \in Q_T$.*

Proof. Fix a point $x \in \Omega$. Without loss of generality assume that $\omega_0(x) > 0$. First suppose $\omega^*(x) > 0$. In the proof of Proposition 2.2 we have shown that $\omega(t,x) \in [\omega^*(x), \omega_0(x)]$ for a.a. $t \in [0,T]$ consequently $\omega(t,x) \ge \min(\omega^*(x), \omega_0(x)) > 0$. Now suppose that $\omega^*(x) = 0$. Define $t^* := \inf\{t > 0 : \omega(t,x) = 0\}$. Then for every $t < t^*$ we have $\omega(t,x) > 0$. By using condition (F1), (F3) it follows that for $\omega > \omega^*(x) = 0$, $f(t,x,\omega,u) \ge -K_1(u)\omega$. Then for a.a. $t \in (0,t^*)$

$$
\omega'(t,x) = f(t,x,\omega(t,x),u(t,x)) \ge -K_1(u(t,x))\omega(t,x).
$$

(Note that ω is absolutely continuous in variable t thus for a.a. $(t,x) \in Q_T$ there exists $\omega'(t,x)$.) By the definition of t^* we can divide by $\omega(t,x)$ and we obtain

$\omega'(t,x)/\omega(t,x) \geq -K_1(u(t,x))$. Observe that the left hand side of the previous inequality equals to $(\log \omega(t,x))'$ thus by integrating the inequality in $(0,t)$ we obtain $\log \omega(t,x) - \log \omega_0(x) \geq -\int_0^t K_1(u(s,x))\,ds$. By taking the exponential of both sides it follows

$$\omega(t,x) \geq \omega_0(x) \cdot e^{-\int_0^t K_1(u(s,x))ds}.$$

From the above estimate it follows that $\omega(t,x) > 0$ a.e. in $[0,T]$. The case $\omega_0(x) < 0$ can be treated the same way. $\square$

Remark 2.6. This proposition shows that if $|\omega_0|$ is a.e. positive and ω_0, ω^* has a.e. the same sign, then for the solution ω of (2.1), $\frac{1}{\omega}$ is a.e. finite. Consequently operator A and B might depend on terms which contain $\frac{1}{\omega}$. The above proof also shows that if the absolute value of the initial value ω_0 is a.e. greater than a positive constant, further, $|\omega^*|$ is greater then a positive lower bound, or K_1 is bounded, then the absolute value of the solution ω of equation (2.1) is also greater than a positive constant a.e. in Q_T thus $\frac{1}{\omega} \in L^\infty(Q_T)$.

Proposition 2.7. *If assumptions (A1)–(A4), (G1)–(G2) hold then for every fixed $\omega \in L^\infty(Q_T)$ and $\mathbf{p} \in X_2$ there exists a solution $u \in D(L)$ of problem (2.2).*

Proof. Since c_1 and γ are continuous functions, for a fixed $\omega \in L^\infty(Q_T)$ functions $c_1(\omega), \gamma(\omega)$ are in $L^\infty(Q_T)$. On the other hand, from $\mathbf{p} \in X_2$ it follows that $|\mathbf{p}|^{\frac{p_2}{q_1}} + |D\mathbf{p}|^{\frac{p_2}{q_1}} \in L^{q_1}(Q_T)$. This means that for fixed $\omega \in L^\infty(Q_T)$ and $\mathbf{p} \in X_2$ conditions (A1)–(A4) are similar to the Léray-Lions conditions for the operator $A(\omega, \cdot, \mathbf{p})\colon X_1 \to X_1^*$. It is not difficult to verify that these conditions imply that operator $A(\omega, \cdot, \mathbf{p})\colon X_1 \to X_1^*$ is bounded, demicontinuous, coercive and pseudomonotone with respect to $D(L)$ (see [2, 4, 7]). In addition, $G(\omega) \in X_1^*$ since by Hölder's inequality and condition (G2)

$$\begin{aligned}
&\Big| \int_{Q_T} g(t,x,\omega(t,x)) v(t,x) dtdx \Big| \\
&\leq \text{const} \cdot \Big(\int_{Q_T} |g(t,x,\omega(t,x))|^{q_1} dtdx \Big)^{1/q_1} \cdot \|v\|_{X_1} \\
&\leq \text{const} \cdot \|c_3(\omega)\|_{L^\infty(Q_T)} \cdot \|k_3\|_{L^{q_1}(Q_T)} \cdot \|v\|_{X_1}.
\end{aligned} \tag{2.8}$$

Then problem $Lu + A(\omega, u, \mathbf{p}) = G(\omega)$ has a solution $u \in D(L)$ for every fixed $\omega \in L^\infty(Q_T)$ and $\mathbf{p} \in X_2$. $\square$

Proposition 2.8. *Under assumptions (H1)–(H2), (B1)–(B4), for every fixed $\omega \in L^\infty(Q_T)$ and $u \in X_1$ problem (2.3) has unique solution $\mathbf{p} \in X_2$.*

Proof. Since $\hat{c_1}, \hat{\gamma}$ are continuous functions, for fixed $\omega \in L^\infty(Q_T)$, the functions $\hat{c}_1(\omega)$ and $\hat{\gamma}(\omega)$ belong to $L^\infty(Q_T)$. Further, $u \in X_1$ implies that $|u|^{\frac{p_1}{q_2}} \in L^{q_2}(Q_T)$ and $|u|^{p_1} \in L^1(Q_T)$. Thus conditions B1-B4 are the Léray-Lions conditions for operator $B(\omega, u, \cdot)\colon X_2 \to X_2^*$. From this it follows that for fixed $\omega \in L^\infty(Q_T)$ and $u \in X_1$ operator $B(\omega, u, \cdot)\colon X_2 \to X_2^*$ is bounded, demicontinuous, coercive and uniformly monotone (see [10]). In addition, $H(\omega, u) \in X_2^*$ because Hölder's inequality and condition (H2) yield

$$\begin{aligned}
&\Big|\int_{Q_T} h(t,x,\omega(t,x),u(t,x))v(t,x)\,dt\,dx\Big| \\
&\le \Big(\int_{Q_T} |h(t,x,\omega(t,x),u(t,x))|^{q_2}dtdx\Big)^{1/q_2}\cdot\|v\|_{L^{p_2}(Q_T)} \qquad (2.9)\\
&\le \text{const}\cdot\|c_4(\omega)\|_{L^\infty(Q_T)}\cdot\Big(\|u\|_{L^{p_1}(Q_T)}^{\frac{p_1}{q_2}}+\|k_4\|_{L^{q_2}}\Big)\cdot\|v\|_{X_2}.
\end{aligned}$$

From the properties of operator $B(\omega,u,\cdot)$ it follows that problem $B(\omega,u,\mathbf{p}) = H(\omega,u)$ has got a unique solution $\mathbf{p}\in X_2$ for every fixed $\omega\in L^\infty(Q_T)$ and $u\in X_1$. □

Now let us turn to the proof of theorem 2.1.

The proof of Theorem 2.1. The idea is the following. We define sequences of approximating solutions of problem (2.1)–(2.3), then we show the boundedness of the sequences. After choosing weakly convergent subsequences we show that the weak limits of the subsequences are solutions of the problem. For simplicity, in the proof we omit the variable (t,x) of functions a_i, b_i if it is not confusing.

Step 1: approximation. Define the sequences $(\omega_k),(u_k),(\mathbf{p}_k)$ by the following. Let $\omega_0(t,x)\equiv u_0(t,x)\equiv\mathbf{p}_0(t,x)\equiv 0$ ($(t,x)\in Q_T$) and for $k=0,1,\dots$ let $\omega_{k+1},u_{k+1},\mathbf{p}_{k+1}$ be a solution of the system

$$\omega_{k+1}(t,x)=\omega_0(x)+\int_0^t f(s,x,\omega_{k+1}(s,x),u_k(s,x))\,ds \qquad (2.10)$$

$$Lu_{k+1}+A(\omega_k,u_{k+1},\mathbf{p}_k)=G(\omega_k) \qquad (2.11)$$

$$B(\omega_k,u_k,\mathbf{p}_{k+1})=H(\omega_k,u_k). \qquad (2.12)$$

By propositions 2.2, 2.7, 2.8, for given $\omega_k,u_k,\mathbf{p}_k$ there exist solutions ω_{k+1} in $L^\infty(Q_T)$, $u_{k+1}\in X_1$ and $\mathbf{p}_{k+1}\in X_2$ of the above system. By the above recurrence we obtain the sequences $(\omega_k)\subset L^\infty(Q_T),(u_k)\subset X_1,(\mathbf{p}_k)\subset X_2$.

Step 2: boundedness. We show that the above defined sequences are bounded. It is obvious that (ω_k) is bounded in $L^\infty(Q_T)$ since by Proposition 2.2 for fixed $\omega_0\in L^\infty(\Omega)$ for the solution of equation (2.10) estimate $\|\omega_{k+1}\|_{L^\infty(Q_T)}\le\|\omega_0\|_{L^\infty(\Omega)}+\|\omega^*\|_{L^\infty(\Omega)}$ holds.

Now let us consider the sequence (u_k). By choosing the test function $v=u_{k+1}$ in (2.11) and by using condition (A4) and the monotonicity of operator L we obtain

$$\begin{aligned}
[G(\omega_k),u_{k+1}] &= [Lu_{k+1},u_{k+1}]+[A(\omega_k,u_{k+1},\mathbf{p}_k),u_{k+1}]\\
&\ge c_2\int_{Q_T}(|u_{k+1}|^{p_1}+|Du_{k+1}|^{p_1}-\gamma(\omega_k)k_2)\\
&\ge c_2\left(\|u_{k+1}\|_{X_1}^{p_1}-\|\gamma(\omega_k)\|_{L^\infty(Q_T)}\cdot\|k_2\|_{L^1(Q_T)}\right)
\end{aligned}$$

where the term $\|\gamma(\omega_k)\|_{L^\infty(Q_T)}\cdot\|k_2\|_{L^1(Q_T)}$ is bounded because of the boundedness of (ω_k) in $L^\infty(Q_T)$. In addition, similarly to (2.8) we have

$$[G(\omega_k),u_{k+1}]\le\text{const}\cdot\|c_3(\omega_k)\|_{L^\infty(Q_T)}\cdot\|k_3\|_{L^{q_1}(Q_T)}\cdot\|u_{k+1}\|_{X_1}. \qquad (2.13)$$

By combining the above two inequalities we obtain

$$\|u_{k+1}\|_{X_1}^{p_1}\le\text{const}\cdot(\|u_{k+1}\|_{X_1}+1)\,.$$

From this inequality, it follows that (u_k) is a bounded sequence in the space X_1.

Let us see now sequence $(\mathbf{p}_k)$. By substituting the test function $v = p_{k+1}$ in (2.12) and by using condition B4 we get

$$[H(\omega_k, u_k), \mathbf{p}_{k+1}] = [B(\omega_k, u_k, \mathbf{p}_{k+1}), \mathbf{p}_{k+1}] \geq \text{const} \cdot \left(\|\mathbf{p}_{k+1}\|_{X_2}^{p_2} - \|u_k\|_{X_1}^{p_1} - 1\right)$$

where the term (as we have shown just before) $\|u_k\|_{X_1}^{p_1}$ is bounded. On the other hand, similarly to estimate (2.9) we have

$$|[H(\omega_k, u_k), \mathbf{p}_{k+1}]| \leq \text{const} \cdot \|\mathbf{p}_{k+1}\|_{X_2} \cdot \|c_4(\omega_k)\|_{L^\infty(Q_T)} \cdot \left(\|u_k\|_{L^{p_1}(Q_T)}^{\frac{p_1}{q_2}} + \|k_4\|_{L^{q_2}(Q_T)}\right).$$

Since (ω_k) is bounded in $L^\infty(Q_T)$ and (u_k) is bounded in X_1, therefore the terms on the right hand side of the above inequality, not containing $\mathbf{p}_{k+1}$, are bounded. Then the previous two estimates yield

$$\|\mathbf{p}_{k+1}\|_{X_2}^{p_2} \leq \text{const} \cdot (\|\mathbf{p}_{k+1}\|_{X_2} + \text{const}) .$$

This means that the sequence $(\mathbf{p}_k)$ is bounded in the space X_2.

We need also the boundedness of the sequence (Lu_k) in X_1^*. By (2.11) it suffices to show that $|[Lu_{k+1}, v]| = |[A(\omega_k, u_{k+1}, \mathbf{p}_k) + G(\omega_k), v]| \leq \text{const} \cdot \|v\|_{X_1}$. By (2.13), $|[G(\omega_k), v]| \leq \text{const} \cdot \|v\|_{X_1}$. In addition, by Hölder's inequality

$$|[A(\omega_k, u_{k+1}, \mathbf{p}_k), v]| \leq \Big(\sum_{i=0}^{n} \|a_i(\omega_k, u_{k+1}, Du_{k+1}, \mathbf{p}_k, D\mathbf{p}_k)\|_{L^{q_1}(Q_T)} \Big) \cdot \|v\|_{X_1}. \tag{2.14}$$

Observe that from condition (A2) it follows that for all i,

$$\begin{aligned} &\|a_i(\omega_k, u_{k+1}, Du_{k+1}, \mathbf{p}_k, D\mathbf{p}_k)\|_{L^{q_1}(Q_T)} \\ &\leq \text{const} \cdot \|c_1(\omega_k)\|_{L^\infty(Q_T)} \left(\|u_{k+1}\|_{X_1}^{p_1} + \|\mathbf{p}_k\|_{X_2}^{p_2} + \|k_1\|_{L^{q_1}(Q_T)}\right). \end{aligned}$$

The right hand side of the above inequality is bounded because of the boundedness of the sequences $(\omega_k), (u_k)$ and $(\mathbf{p}_k)$ (and by the continuity of the function c_1). Combining this with estimate (2.14) we obtain the desired estimate $|[A(\omega_k, u_{k+1}, \mathbf{p}_k), v]| \leq \text{const} \cdot \|v\|_{X_1}$ thus (Lu_k) is a bounded sequence in the space X_1^*.

Step 3: convergence. In the preceding step we showed that the sequences $(u_k), (Lu_k), (\mathbf{p}_k)$ are bounded (in reflexive Banach spaces) therefore each has a weakly convergent subsequence (which will be denoted as the original sequence for simplicity), so there exist $u \in X_1$, $w \in X_1^*$ and $\mathbf{p} \in X_2$ such that

$$\begin{gathered} u_k \to u \quad \text{weakly in } X_1; \\ Lu_k \to w \quad \text{weakly in } X_1^*; \\ \mathbf{p}_k \to \mathbf{p} \quad \text{weakly in } X_2. \end{gathered}$$

Further, from the properties of operator L (see [10]) it follows that $w \in D(L)$ and $w = Lu$. Thus by applying the well known embedding theorem (see [7]) it follows that there exists a subsequence of (u_k) which is convergent in $L^{p_1}(Q_T)$ hence it has got an a.e. convergent subsequence. In what follows, we use these subsequences, that is, we suppose that

$$\begin{gathered} u_k \to u \quad \text{weakly in } X_1; \\ u_k \to u \quad \text{in } L^{p_1}(Q_T); \\ u_k \to u \quad \text{a.e. in } Q_T; \\ Lu_k \to Lu \quad \text{weakly in } X_1^*; \\ \mathbf{p}_k \to \mathbf{p} \quad \text{weakly in } X_2. \end{gathered}$$

Now we show that $\omega, u, \mathbf{p}$ are solutions of problem (2.1)–(2.3).

We start with equation (2.10). Since $u_k \to u$ in $L^{p_1}(Q_T)$, further, ω_{k+1} is the solution of equation (2.10), by Proposition 2.3 it follows that $\omega_k \to \omega$ a.e. in Q_T and functions ω, u satisfy the integral equation (2.1).

Now let us consider equation (2.12). First we show that $\mathbf{p}_k \to \mathbf{p}$ in X_2. By condition B3 we have

$$[B(\omega_k, u_k, \mathbf{p}_{k+1}) - B(\omega_k, u_k, \mathbf{p}), \mathbf{p}_{k+1} - \mathbf{p}] \geq \hat{C} \cdot \|\mathbf{p}_{k+1} - \mathbf{p}\|_{X_2}^{p_2}. \tag{2.15}$$

On the other hand,

$$\begin{aligned} &[B(\omega_k, u_k, \mathbf{p}_{k+1}) - B(\omega_k, u_k, \mathbf{p}), \mathbf{p}_{k+1} - \mathbf{p}] \\ &= [B(\omega_k, u_k, \mathbf{p}_{k+1}), \mathbf{p}_{k+1} - \mathbf{p}] + [B(\omega, u, \mathbf{p}) - B(\omega_k, u_k, \mathbf{p}), \mathbf{p}_{k+1} - \mathbf{p}] \\ &- [B(\omega, u, \mathbf{p}), \mathbf{p}_{k+1} - \mathbf{p}], \end{aligned} \tag{2.16}$$

and we show that each term of the right hand side tends to 0. The last term evidently converges to 0 since $(\mathbf{p}_k)$ is weakly convergent in X_2. In order to verify the convergence of the second term, observe that

$$\begin{aligned} &|[B(\omega_k, u_k, \mathbf{p}) - B(\omega, u, \mathbf{p}), \mathbf{p}_{k+1} - \mathbf{p}]| \\ &\leq \sum_{i=0}^{n} \|b_i(\omega_k, u_k, \mathbf{p}, D\mathbf{p}) - b_i(\omega, u, \mathbf{p}, D\mathbf{p})\|_{L^{q_2}(Q_T)} \cdot \|\mathbf{p}_{k+1} - \mathbf{p}\|_{X_2} \end{aligned} \tag{2.17}$$

and by condition (B2)

$$\begin{aligned} &|b_i(\omega_k, u_k, \mathbf{p}, D\mathbf{p}) - b_i(\omega, u, \mathbf{p}, D\mathbf{p})|^{q_2} \\ &\leq \text{const} \cdot (|\hat{c_1}(\omega_k)|^{q_2} + |\hat{c_1}(\omega)|^{q_2}) \Big(|\mathbf{p}|^{p_2} + |D\mathbf{p}|^{p_2} + |u_k|^{p_1} + |u|^{p_1} + |\hat{k}_1|^{q_2} \Big). \end{aligned}$$

Since (ω_k) is bounded in $L^\infty(Q_T)$ and (u_k) is convergent in $L^{p_1}(Q_T)$, therefore the right hand side of the above inequality is equi-integrable (see [4]) hence the left hand side is equi-integrable, too. In addition, the left hand side a.e. converges to 0 (because of the a.e. convergence of (ω_k) and (u_k)), therefore by Vitali's theorem the left hand side converges in $L^1(Q_T)$ to the zero function. From this (and from the boundedness of $(\mathbf{p}_k)$) it follows that the right hand side of (2.17) tends to 0. By recurrence (2.12) we have $B(\omega_k, u_k, \mathbf{p}_{k+1}) = H(\omega_k, u_k)$, further, $\mathbf{p}_{k+1} \to \mathbf{p}$ weakly in X_2 hence in order to prove the convergence of the first term of the right hand side of (2.16) it suffices to show that $H(\omega_k, u_k) \to H(\omega, u)$ in X_2^*. By Hölder's inequality

$$
\begin{aligned}
&|[H(\omega_k, u_k) - H(\omega, u), v]| \\
&\le \Big(\int_{Q_T} |h(t, x, \omega_k(t, x), u_k(t, x)) - h(t, x, \omega(t, x), u(t, x))|^{q_2}\, dtdx\Big)^{1/q_2} \cdot \|v\|_{X_2} \\
&= \Big(\int_{Q_T} b_k(t, x)\, dt\, dx\Big)^{1/q_2} \cdot \|v\|_{X_2}.
\end{aligned}
$$

It is clear from the above inequality that $\|H(\omega_k, u_k) - H(\omega, u)\|_{X_2^*} \le \|b_k\|_{L^1(Q_T)}^{1/q_2}$ so we only have to prove that (b_k) converges to 0 in $L^1(Q_T)$. By condition (H2) we obtain

$$
|b_k| \le \text{const} \cdot (|c_4(\omega_k)|^{q_2} + |c_4(\omega)|^{q_2}) (|u_k|^{p_1} + |u|^{p_1} + |k_4|^{q_2}) .
$$

The right hand side of the above inequality is equi-integrable in $L^1(Q_T)$ (because of the convergence of (u_k) in $L^{p_1}(Q_T)$ and the boundedness of (ω_k) in $L^\infty(Q_T)$) thus (b_k) is equi-integrable, too. Besides, (b_k) a.e. converges to 0 since $u_k \to u$ and $\omega_k \to \omega$ a.e. in Q_T and H is continuous in these variables. Then by Vitali's theorem (b_k) tends to 0 in $L^1(Q_T)$. From the above arguments it follows that $H(\omega_k, u_k) \to H(\omega, u)$ in X_2^* so the right hand side of the equation (2.16) converges to 0 thus (2.15) implies that $\mathbf{p}_{k+1} \to \mathbf{p}$ in X_2. This means that $\mathbf{p}_{k+1} \to \mathbf{p}$ in $L^{p_2}(Q_T)$, too, so we may assume that $\mathbf{p}_{k+1} \to \mathbf{p}$ a.e. in Q_T.

Now we show that $B(\omega_k, u_k, \mathbf{p}_{k+1}) \to B(\omega, u, \mathbf{p})$ weakly in X_2^*. Then from recurrence (2.12) we obtain $B(\omega, u, \mathbf{p}) = H(\omega, u)$ (we have seen earlier that $H(\omega_k, u_k) \to H(\omega, u)$ weakly in X_2^*) i.e. $\omega, u, \mathbf{p}$ are solutions of problem (2.3). In order to verify the weak convergence $B(\omega_k, u_k, \mathbf{p}_{k+1}) \to B(\omega, u, \mathbf{p})$ observe that

$$
\begin{aligned}
&|[B(\omega_k, u_k, \mathbf{p}_{k+1}) - B(\omega, u, \mathbf{p}), v]| \\
&\le \sum_{i=0}^{n} \|b_i(\omega_k, u_k, \mathbf{p}_{k+1}, D\mathbf{p}_{k+1}) - b_i(\omega, u, \mathbf{p}, D\mathbf{p})\|_{L^{q_2}(Q_T)} \cdot \|v\|_{X_2},
\end{aligned} \tag{2.18}
$$

and by condition (B2)

$$
\begin{aligned}
&|b_i(\omega_k, u_k, \mathbf{p}_{k+1}, D\mathbf{p}_{k+1}) - b_i(\omega, u, \mathbf{p}, D\mathbf{p})|^{q_2} \\
&\le \text{const} \cdot |\hat{c_1}(\omega_k)|^{q_2} \Big(|\mathbf{p}_{k+1}|^{p_2} + |D\mathbf{p}_{k+1}|^{p_2} + |u_k|^{p_1} + |\hat{k}_1|^{q_2}\Big) \\
&\quad + \text{const} \cdot |\hat{c_1}(\omega)|^{q_2} \Big(|\mathbf{p}|^{p_2} + |D\mathbf{p}|^{p_2} + |u|^{p_1} + |\hat{k}_1|^{q_2}\Big) .
\end{aligned}
$$

The right hand side of the above inequality is equi-integrable in $L^1(Q_T)$ (because of the strong convergence of $(\mathbf{p}_k)$ in X_2 and the boundedness of $(\hat{c}_1(\omega_k))$ in $L^\infty(Q_T)$) thus the left hand side is also equi-integrable. Moreover, by the Carathéodory conditions the left hand side a.e. converges to the zero function thus from Vitali's theorem it follows that the right hand side of (2.18) tends to 0. Consequently $B(\omega_k, u_k, \mathbf{p}_{k+1}) - B(\omega, u, \mathbf{p}) \to 0$ weakly in X_2^*.

In the case of equation (2.11) we can apply the same argument as in the case of equation (2.12). We have already shown that $Lu_{k+1} \to Lu$ weakly in X_1^*. Now we verify that $G(\omega_k) \to G(\omega)$ in X_1^* and $A(\omega_k, u_{k+1}, \mathbf{p}_k) \to A(\omega, u, \mathbf{p})$ weakly in X_1^* then these convergences from recurrence (2.11) yield (2.2). To the strong convergence $G(\omega_k) \to G(\omega)$ observe that by Hölder's inequality

$$
|[G(\omega_k) - G(\omega), v]| \le \|g(\cdot, \omega_k) - g(\cdot, \omega)\|_{L^{q_1}(Q_T)} \cdot \|v\|_{X_1}
$$

which implies $\|G(\omega_k)-G(\omega)\|_{X_1^*} \le \|g(\cdot,\omega_k)-g(\cdot,\omega)\|_{L^{q_1}(Q_T)}$. From condition (G2) and the boundedness of (ω_k) we have

$$\begin{aligned}&|g(t,x,\omega_k(t,x)) - g(t,x,\omega(t,x))|\\&\le \text{const}\cdot \big(\|c_3(\omega_k)\|_{L^\infty(Q_T)} + \|c_3(\omega)\|_{L^\infty(Q_T)}\big)\cdot |k_3(t,x)|\end{aligned}$$

thus the left hand side has a majorant in $L^{q_1}(Q_T)$ hence by the a.e. convergence of (ω_k) and the continuity of G, Lebesgue's theorem yields the convergence of the left hand side to 0 in $L^{q_1}(Q_T)$.

For the weak convergence $A(\omega_k, u_{k+1}, \mathbf{p}_k) \to A(\omega, u, \mathbf{p})$ we first show that $u_k \to u$ in X_1. To this end, it suffices to show that $Du_k \to Du$ in $L^{p_1}(Q_T)$ since we have verified so far that $u_k \to u$ in $L^{p_1}(Q_T)$. By the monotonicity of operator L,

$$\begin{aligned}&[Lu_{k+1} - Lu, u_{k+1} - u] + [A(\omega_k, u_{k+1}, \mathbf{p}_k) - A(\omega_k, u, \mathbf{p}_k), u_{k+1} - u]\\&\ge \sum_{i=1}^n \int_{Q_T} \big(a_i(\omega_k, u_{k+1}, Du_{k+1}, \mathbf{p}_k, D\mathbf{p}_k) - a_i(\omega_k, u_{k+1}, Du, \mathbf{p}_k, D\mathbf{p}_k)\big)\\&\quad\times \big(D_i u_{k+1} - D_i u\big)\\&\quad+ \sum_{i=1}^n \int_{Q_T} \big(a_i(\omega_k, u_{k+1}, Du, \mathbf{p}_k, D\mathbf{p}_k) - a_i(\omega_k, u, Du, \mathbf{p}_k, D\mathbf{p}_k)\big)(D_i u_{k+1} - D_i u)\\&\quad+ \int_{Q_T} \big(a_0(\omega_k, u_{k+1}, Du_{k+1}, \mathbf{p}_k, D\mathbf{p}_k) - a_0(\omega_k, u, Du, \mathbf{p}_k, D\mathbf{p}_k)\big)(u_{k+1} - u).\end{aligned}\tag{2.19}$$

Observe that by condition (A3) the first term on the right hand side of the above inequality is greater than $C\cdot \|Du_{k+1} - Du\|_{L^{p_1}(Q_T)}$. We show that the left hand side and the integrals on the right hand side converge to 0, then the convergence of (Du_k) in $L^{p_1}(Q_T)$ immediately follows. Consider the decomposition

$$\begin{aligned}&[Lu_{k+1} - Lu, u_{k+1} - u] + [A(\omega_k, u_{k+1}, \mathbf{p}_k) - A(\omega_k, u, \mathbf{p}_k), u_{k+1} - u]\\&= [Lu_{k+1} + A(\omega_k, u_{k+1}, \mathbf{p}_k), u_{k+1} - u] - [Lu, u_{k+1} - u] - [A(\omega_k, u, \mathbf{p}_k), u_{k+1} - u].\end{aligned}$$

The first term on the right hand side equals to $[G(\omega_k), u_{k+1} - u]$ because of the recurrence (2.11). By the strong convergence of $(G(\omega_k))$ and the weak convergence of (u_k) it follows that $[G(\omega_k), u_{k+1} - u] \to 0$. The second term tends to 0 since (u_k) is weakly convergent. By condition (A2), the a.e. convergence of (ω_k), the strong convergence of $(\mathbf{p}_k)$ it is easy to see (similar to the case of operator B, see (2.17)) that the third term also tends to 0 which yields the convergence of the left hand side of (2.19) to 0. By Hölder's inequality

$$\begin{aligned}&\Big|\int_{Q_T} \left(a_i(\omega_k, u_{k+1}, Du, \mathbf{p}_k, D\mathbf{p}_k) - a_i(\omega_k, u, Du, \mathbf{p}_k, D\mathbf{p}_k)\right)(D_i u_{k+1} - D_i u)\Big|\\&\le \|a_i(\omega_k, u_{k+1}, Du, \mathbf{p}_k, D\mathbf{p}_k) - a_i(\omega_k, u, Du, \mathbf{p}_k, D\mathbf{p}_k)\|_{L^{q_1}(Q_T)}\\&\quad\times \|D_i u_{k+1} - D_i u\|_{L^{p_1}(Q_T)}\end{aligned}$$

where the coefficient of the bounded term $\|D_i u_{k+1} - D_i u\|_{L^{p_1}(Q_T)}$ converges to 0 by Vitali's theorem. In fact, $(a_i(\omega_k, u_{k+1}, Du, \mathbf{p}_k, D\mathbf{p}_k) - a_i(\omega_k, u, Du, \mathbf{p}_k, D\mathbf{p}_k)) \to 0$ a.e. in Q_T, further,

$$\begin{aligned}&|a_i(\omega_k, u_{k+1}, Du, \mathbf{p}_k, D\mathbf{p}_k) - a_i(\omega_k, u, Du, \mathbf{p}_k, D\mathbf{p}_k)|^{q_1}\\&\le \text{const}\cdot |c_1(\omega_k)|\cdot (|u_{k+1}|^{p_1} + |u|^{p_1} + |\mathbf{p}_k|^{p_2} + |k_1|^{q_1})\end{aligned}$$

where the right hand side converges in $L^1(Q_T)$. In order to verify the convergence of the last integral on the right hand side of (2.19), we use Hölder's inequality and condition (A2) and obtain

$$\begin{aligned}
&\Big|\int_{Q_T}(a_0(\omega_k,u_{k+1},Du_{k+1},\mathbf{p}_k,D\mathbf{p}_k)-a_0(\omega_k,u,Du,\mathbf{p}_k,D\mathbf{p}_k))\,(u_{k+1}-u)\Big|\\
&\le \text{const}\cdot\|c_1(\omega_k)\|_{L^\infty(Q_T)}\cdot\Big(\|u_{k+1}\|_{X_1}^{\frac{p_1}{q_1}}+\|u\|_{X_1}^{\frac{p_1}{q_1}}+\|\mathbf{p}_k\|_{X_2}^{\frac{p_2}{q_1}}+\|k_1\|_{L^{q_1}(Q_T)}\Big)\\
&\quad\times\|u_{k+1}-u\|_{L^{p_1}(Q_T)}.
\end{aligned}$$

By the strong convergence of $(\mathbf{p}_k)$ in X_2 and (u_k) in $L^{p_1}(Q_T)$ and by the boundedness of (u_k) in X_1 the right hand side tends to 0.

Now the weak convergence $A(\omega_k,u_{k+1},\mathbf{p}_k)\to A(\omega,u,\mathbf{p})$ in X_1^* follows easily by condition (A2), the strong convergence and by Vitali's theorem (the same as in the case of operator B). So we have shown that $\omega,u,\mathbf{p}$ are solutions of problem (2.2).

Summarizing, we have verified that $\omega,u,\mathbf{p}$ are solutions of system (2.1)–(2.3) hence the proof of the theorem is complete. □

Remark 2.9. From the above proof it is clear that if we suppose

(A3') There exists a constant $C>0$ such that for a.a. $(t,x)\in Q_T$ and every $(\omega,\zeta_0,\zeta,\mathbf{p},D\mathbf{p}),(\omega,\eta_0,\eta,\mathbf{p},D\mathbf{p})\in\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$

$$\begin{aligned}
&\sum_{i=0}^{n}(a_i(t,x,\omega,\zeta_0,\zeta,\mathbf{p},D\mathbf{p})-a_i(t,x,\omega,\eta_0,\eta,\mathbf{p},D\mathbf{p}))\,(\zeta_i-\eta_i)\\
&\ge C\cdot(|\zeta_0-\eta_0|^{p_1}+|\zeta-\eta|^{p_1})
\end{aligned}$$

instead of A3, then the theorem remains true. Indeed, it simplifies equation (2.19), on the right hand side will stand $|\zeta_0-\eta_0|^{p_1}+|\zeta-\eta|^{p_1}$.

3. Examples

In this section we give some examples of functions a_i,b_i $(i=0,\dots,n)$ that fulfil conditions (A1)–(A4), (B1)–(B4). Let us start with a general example. Suppose that functions a_i,b_i have the form

$$\begin{aligned}
a_i(t,x,\omega,\zeta_0,\zeta,\mathbf{p},D\mathbf{p})&=(P(\omega)+\mathcal{P}(\omega)Q(\mathbf{p},D\mathbf{p}))\,\alpha_i(t,x,\zeta)\\
&\quad+\Big(\tilde P(\omega)+\tilde{\mathcal{P}}(\omega)\tilde Q(\mathbf{p},D\mathbf{p})\Big)\tilde\alpha_i(t,x,\zeta),\quad\text{for } i\ne 0,
\end{aligned}\tag{3.1}$$

$$\begin{aligned}
a_0(t,x,\omega,\zeta_0,\zeta,\mathbf{p},D\mathbf{p})&=(P(\omega)+\mathcal{P}(\omega)Q(\mathbf{p},D\mathbf{p}))\,\alpha_0(t,x,\zeta_0,\zeta)\\
&\quad+\Big(\tilde P_0(\omega)+\tilde{\mathcal{P}}(\omega)\tilde Q(\mathbf{p},D\mathbf{p})\Big)\,\tilde\alpha_0(t,x,\zeta_0,\zeta),
\end{aligned}\tag{3.2}$$

$$\begin{aligned}
b_i(t,x,\omega,u,\zeta_0,\zeta)&=(R(\omega)+\mathcal{R}(\omega)S(u))\beta_i(t,x,\zeta_0,\zeta)\\
&\quad+\Big(\tilde R(\omega)+\tilde{\mathcal{R}}(\omega)\tilde S(u)\Big)\,\tilde\beta_i(t,x,\zeta_0,\zeta),\quad i=0,\dots,n
\end{aligned}\tag{3.3}$$

where the following hold.

(E1) Functions $\alpha_i,\tilde\alpha_i,\beta_i,\tilde\beta_i\colon Q_T\times\mathbb{R}^n\to\mathbb{R}$ $(i=1,\dots,n)$ are Carathéodory functions, i.e. they are measurable in $(t,x)\in Q_T$ for every $\zeta\in\mathbb{R}^n$ and continuous in $\zeta\in\mathbb{R}^n$ for a.a. $(t,x)\in Q_T$. Functions $\alpha_0,\tilde\alpha_0\colon Q_T\times\mathbb{R}^{n+1}\to\mathbb{R}$ are Carathéodory functions, i.e. they are measurable in $(t,x)\in Q_T$ for every $(\zeta_0,\zeta)\in\mathbb{R}^{n+1}$ and continuous in $(\zeta_0,\zeta)\in\mathbb{R}^{n+1}$ for a.a. $(t,x)\in Q_T$.

(E2) There exist constants $c_1, \hat{c}_1 > 0$, $0 \le r_1 < p_1 - 1$, $0 \le r_2 < p_2 - 1$ and functions $k_1 \in L^{q_1}(Q_T)$, $\hat{k}_1 \in L^{q_2}(\Omega)$ such that
  (a) $|\alpha_i(t,x,\zeta)| \le c_1|\zeta|^{p_1-1} + k_1(t,x)$, if $i \neq 0$,
  (b) $|\alpha_0(t,x,\zeta_0,\zeta)| \le c_1\left(|\zeta_0|^{p_1-1} + |\zeta|^{p_1-1}\right) + k_1(t,x)$,
  (c) $|\tilde{\alpha}_i(t,x,\zeta)| \le c_1|\zeta|^{r_1}$, if $i \neq 0$,
  (d) $|\tilde{\alpha}_0(t,x,\zeta_0,\zeta)| \le c_1\left(|\zeta_0|^{r_1} + |\zeta|^{r_1}\right)$,
  (e) $|\beta_i(t,x,\zeta_0,\zeta)| \le \hat{c}_1\left(|\zeta_0|^{p_2-1} + |\zeta|^{p_2-1}\right) + \hat{k}_1(t,x)$, if $i \neq 0$,
  (f) $|\tilde{\beta}_i(t,x,\zeta_0,\zeta)| \le \hat{c}_1\left(|\zeta_0|^{r_2} + |\zeta|^{r_2}\right)$, if $i \neq 0$
  for a.a. $(t,x) \in Q_T$ and every $(\zeta_0,\zeta) \in \mathbb{R}^{n+1}$ $(i = 0,\dots,n)$.

(E3) There exist constants $C, \hat{C} > 0$ such that for a.a. $(t,x) \in Q_T$ and every $(\zeta_0,\eta_0), (\zeta_0,\eta) \in \mathbb{R}^{n+1}$
  (a) $\sum_{i=1}^n \left(\alpha_i(t,x,\zeta) - \alpha_i(t,x,\eta)\right)(\zeta_i - \eta_i) \ge C \cdot |\zeta - \eta|^{p_2}$,
  (b) $\sum_{i=1}^n \left(\tilde{\alpha}_i(t,x,\zeta) - \tilde{\alpha}_i(t,x,\eta)\right)(\zeta_i - \eta_i) \ge 0$,
  (c) $\sum_{i=0}^n \left(\beta_i(t,x,\zeta_0,\zeta) - \beta_i(t,x,\eta_0,\eta)\right)(\zeta_i - \eta_i)$
  $\ge \hat{C} \cdot \left(|\zeta_0 - \eta_0|^{p_2} + |\zeta - \eta|^{p_2}\right)$,
  (d) $\sum_{i=0}^n \left(\tilde{\beta}_i(t,x,\zeta_0,\zeta) - \tilde{\beta}_i(t,x,\eta_0,\eta)\right)(\zeta_i - \eta_i) \ge 0$.

(E4) There exist constants $c_2, \hat{c}_2 > 0$ and functions $k_2, \hat{k}_2 \in L^1(Q_T)$ such that
  (a) $\sum_{i=1}^n \alpha_i(t,x,\zeta)\zeta_i + \alpha_0(t,x,\zeta_0,\zeta)\zeta_0 \ge c_2(|\zeta_0|^{p_1} + |\zeta|^{p_1}) - k_2(t,x)$,
  (b) $\sum_{i=1}^n \tilde{\alpha}_i(t,x,\zeta)\zeta_i + \tilde{\alpha}_0(t,x,\zeta_0,\zeta)\zeta_0 \ge 0$
  (c) $\sum_{i=0}^n \beta_i(t,x,\zeta_0,\zeta)\zeta_i \ge \hat{c}_2(|\zeta_0|^{p_2} + |\zeta|^{p_2}) - \hat{k}_2(t,x)$,
  (d) $\sum_{i=1}^n \tilde{\beta}_i(t,x,\zeta_0,\zeta)\zeta_i \ge 0$
  for a.a. $(t,x) \in Q_T$ and every $(\zeta_0,\zeta) \in \mathbb{R}^{n+1}$.

(E5) (a) Functions $P, \mathcal{P}, \tilde{P}, \tilde{\mathcal{P}}, \tilde{P}_0 \colon \mathbb{R} \to \mathbb{R}$, $Q, \tilde{Q} \colon \mathbb{R}^{n+1} \to \mathbb{R}$ are continuous, $\tilde{P}(\omega) + \tilde{\mathcal{P}}(\omega)\tilde{Q}(\mathbf{p}, D\mathbf{p}) \ge 0$ and there exists a constant $c > 0$ such that $P(\omega) + \mathcal{P}(\omega)Q(\mathbf{p}, D\mathbf{p}) \ge c$ for every $\omega \in \mathbb{R}$, $(\mathbf{p}, D\mathbf{p}) \in \mathbb{R}^{n+1}$. Further, Q is bounded, $|\tilde{Q}(\mathbf{p}, D\mathbf{p})| \le \text{const} \cdot \left(|\mathbf{p}|^{\frac{p_2(p_1-1-r_1)}{p_1}} + |D\mathbf{p}|^{\frac{p_2(p_1-1-r_1)}{p_1}}\right)$ where constant r_1 is given in (E2).
  (b) Functions $R, \mathcal{R}, \tilde{R}, \tilde{\mathcal{R}}, S, \tilde{S} \colon \mathbb{R} \to \mathbb{R}$ are continuous, $\tilde{R}(\omega) + \tilde{\mathcal{R}}(\omega)\tilde{S}(u) \ge 0$ and there exists a positive constant c such that $R(\omega) + \mathcal{R}(\omega)S(u) \ge c$ for every $\omega \in \mathbb{R}$ and $u \in \mathbb{R}$, further, S is bounded and $|\tilde{S}(u)| \le \text{const} \cdot |u|^{\frac{p_1(p_2-1-r_2)}{p_2}}$ where constant r_2 is given in (E2).

Proposition 3.1. *If assumptions (E1)–(E5) hold then functions* (3.1)–(3.3) *fulfils conditions (A1)–(A4), (B1)–(B4).*

Proof. We verify only conditions (A1)–(A4), the other can be shown by using similar arguments. From (E1) immediately follows condition (A1). In order to obtain (A2), let us apply Young's inequality with exponents $p^* = \frac{p_1-1}{r_1}$ and $q^* = \frac{p_1-1}{p_1-1-r_1}$ and use the growth condition imposed on $\tilde{\alpha}_i$ and $\tilde{Q}$. We obtain

$$
\begin{aligned}
&\left|\left(\tilde{P}(\omega) + \tilde{\mathcal{P}}(\omega)\tilde{Q}(\mathbf{p}, D\mathbf{p})\right)\tilde{\alpha}_i(t,x,\zeta)\right| \\
&\le \text{const} \cdot \left(|\tilde{P}(\omega)|^{q^*} + |\tilde{\mathcal{P}}(\omega)|^{q^*}|\tilde{Q}(\mathbf{p}, D\mathbf{p})|^{q^*} + \left(|\zeta|^{r_1}\right)^{\frac{p_1-1}{r_1}}\right) \\
&\le \text{const} \cdot \left(|\tilde{P}(\omega)|^{q^*} + |\tilde{\mathcal{P}}(\omega)|^{q^*} \cdot \left(|\mathbf{p}|^{\frac{p_2}{q_1}} + |D\mathbf{p}|^{\frac{p_2}{q_1}}\right) + |\zeta|^{p_1-1}\right).
\end{aligned}
$$

In addition,

$$
\left|\left(P(\omega) + \mathcal{P}(\omega)Q(\mathbf{p}, D\mathbf{p})\right)\alpha_i(t,x,\zeta)\right| \le \text{const} \cdot \left(|P(\omega) + |\mathcal{P}(\omega)|\right) \cdot \left(|\zeta|^{p_1} + |k_1(t,x)|\right).
$$

Then using condition (E2) easily follows that

$$|a_i(t,x,\omega,\zeta_0,\zeta,\mathbf{p},D\mathbf{p})| \le \text{const}\cdot\Big(|P(\omega)| + |\mathcal{P}(\omega) + |\tilde{P}(\omega)|^{q^*} + |\tilde{\mathcal{P}}(\omega)|^{q^*} + 1\Big) \\ \times \Big(|\zeta|^{p_1-1} + |k_1(t,x)| + |\mathbf{p}|^{\frac{p_2}{q_1}} + |D\mathbf{p}|^{\frac{p_2}{q_1}}\Big),$$

and similarly holds for $i=0$. This means that condition (A2) holds.

Now by the nonnegativity of $\tilde{P}+\tilde{\mathcal{P}}\tilde{Q}$, the uniform positivity of the sum $P+\mathcal{P}Q$, and by condition E3,

$$\begin{aligned}
&\sum_{i=1}^n \left(a_i(t,x,\omega,\zeta_0,\zeta,\mathbf{p},D\mathbf{p}) - a_i(t,x,\omega,\zeta_0,\eta,\mathbf{p},D\mathbf{p})\right)(\zeta_i-\eta_i)\\
&= (P(\omega)+\mathcal{P}(\omega)Q(\mathbf{p},D\mathbf{p}))\sum_{i=1}^n \left(\alpha_i(t,x,\zeta)-\alpha_i(t,x,\eta)\right)(\zeta_i-\eta_i)\\
&\quad + \Big(\tilde{P}(\omega)+\tilde{\mathcal{P}}(\omega)\tilde{Q}(\mathbf{p},D\mathbf{p})\Big)\sum_{i=1}^n \left(\tilde{\alpha}_i(t,x,\zeta)-\tilde{\alpha}_i(t,x,\eta)\right)(\zeta_i-\eta_i)\\
&\ge c\cdot|\zeta-\eta|^{p_1},
\end{aligned}$$

hence condition (A3) also holds.

Conditions (E4) and (E5) yield

$$\begin{aligned}
&\sum_{i=0}^n a_i(t,x,\omega,\zeta_0,\zeta,\mathbf{p},D\mathbf{p})\zeta_i\\
&= (P(\omega)+\mathcal{P}(\omega)Q(\mathbf{p},D\mathbf{p}))\cdot\Big(\sum_{i=1}^n \alpha_i(t,x,\zeta)\zeta_i + \alpha_0(t,x,\zeta_0,\zeta)\zeta_0\Big)\\
&\quad + \Big(\tilde{P}(\omega)+\tilde{\mathcal{P}}(\omega)\tilde{Q}(\mathbf{p},D\mathbf{p})\Big)\cdot\Big(\sum_{i=1}^n \tilde{\alpha}_i(t,x,\zeta)\zeta_i + \tilde{\alpha}_0(t,x,\zeta_0,\zeta)\zeta_0\Big)\\
&\quad + \Big(\tilde{P}_0(\omega)-\tilde{P}(\omega)\Big)\tilde{\alpha}_0(t,x,\zeta_0,\zeta)\zeta_0\\
&\ge c\cdot\left(c_2\left(|\zeta_0|^{p_1}+|\zeta|^{p_1}\right) - k_2(t,x)\right) + \Big(\tilde{P}_0(\omega)-\tilde{P}(\omega)\Big)\tilde{\alpha}_0(t,x,\zeta_0,\zeta)\zeta_0
\end{aligned}$$

By applying Young's inequality two times and by condition (E2) we obtain

$$\begin{aligned}
&\Big|\big(\tilde{P}_0(\omega)-\tilde{P}(\omega)\big)\tilde{\alpha}_0(t,x,\zeta_0,\zeta)\zeta_0\Big|\\
&\le \text{const}\cdot\Big(\Big|\text{const}(\varepsilon)\Big(|\tilde{P}_0(\omega)|+|\tilde{P}(\omega)|\Big)\tilde{\alpha}_0(t,x,\zeta_0,\zeta)\Big|^{q_1} + \varepsilon^{p_1}|\zeta_0|^{p_1}\Big)\\
&\le \text{const}\cdot\Big(\text{const}(\varepsilon)\Big||\tilde{P}_0(\omega)|+|\tilde{P}(\omega)|\Big|^{q^*q_1} + \varepsilon^{p_1}|\zeta_0|^{p_1} + \varepsilon^{p_1}|\zeta|^{p_1}\Big).
\end{aligned}$$

By choosing sufficiently small $\varepsilon>0$ we obtain from the above two estimates that

$$\begin{aligned}
&\sum_{i=0}^n a_i(t,x,\omega,\zeta_0,\zeta,\mathbf{p},D\mathbf{p})\zeta_i\\
&\ge \text{const}\cdot\Big(|\zeta_0|^{p_1}+|\zeta|^{p_1} - k_2(t,x) - \Big||\tilde{P}_0(\omega)|+|\tilde{P}(\omega)|\Big|^{q^*q_1}\Big)
\end{aligned}$$

thus condition (A4) is fulfilled. □

The simplest and most applied examples for functions α_i, $\tilde{\alpha}_i$, β_i, $\tilde{\beta}_i$ are the following:

$$\alpha_i(t,x,\zeta) = \zeta_i|\zeta|^{p_1-2} \quad (i \neq 0), \quad \beta_i(t,x,\zeta_0,\zeta) = \zeta_i|\zeta|^{p_2-2} \quad (i \neq 0),$$
$$\alpha_0(t,x,\zeta_0,\zeta) = \zeta_0|\zeta_0|^{p_1-2}, \quad \beta_0(t,x,\zeta_0,\zeta) = \zeta_0|\zeta_0|^{p_2-2},$$
$$\tilde{\alpha}_i(t,x,\zeta) = \zeta_i|\zeta|^{r_1-1} \quad (i \neq 0), \quad \tilde{\beta}_i(t,x,\zeta_0,\zeta) = \zeta_i|\zeta|^{r_2-1} \quad (i \neq 0),$$
$$\tilde{\alpha}_0(t,x,\zeta_0,\zeta) = \zeta_0|\zeta_0|^{r_1-1}, \quad \tilde{\beta}_0(t,x,\zeta_0,\zeta) = \zeta_0|\zeta_0|^{r_2-1}.$$

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