On a nonlinear system containing nonlocal terms related to a fluid flow model

Ádám Besenyei

Abstract

We consider a nonlinear system of differential equations where the main parts may contain nonlocal dependence on the unknowns. This system is a generalization of a model describing fluid flow in porous medium. Existence of weak solutions, boundedness and stabilization of solutions as \( t \to \infty \) is shown by using the theory of monotone operators, and some examples are given.

1 Introduction

This paper was motivated by works [3, 4, 6, 13]. In [13] parabolic quasilinear functional differential equations were investigated where also the main part contains functional dependence on the unknown function, for example the integral of it. Such problems may occur, e.g., in a diffusion process for heat or population where the diffusion coefficient depends on nonlocal quantity. In [13] existence of weak solutions in time interval \((0,T)\) \((0<T<\infty)\), boundedness and stabilization of solutions as \( t \to \infty \) were proved for that type of parabolic problems. Some of these results were extended to systems of equations and to parabolic variational inequalities in \([3, 5]\). In this paper we extend these results to a system which was motivated by the following one-dimensional model describing fluid flow (carrying chemical species) in porous medium that was studied in \([12]\):

\[
\begin{align*}
\omega(t,x)u_t(t,x) &= \alpha \cdot (|v(t,x)|u_x(t,x))_x + K(\omega(t,x))p_x(t,x)u_x(t,x) - ku(t,x)g(\omega(t,x)) \quad (1) \\
\omega_1(t,x) &= bu(t,x)g(\omega(t,x)) \quad (2) \\
(K(\omega(t,x))p_x(t,x))_x &= bu(t,x)g(\omega(t,x)), \quad (3) \\
v(t,x) &= -K(\omega(t,x))p_x(t,x), \quad t > 0, \ x \in (0,1), \quad (4)
\end{align*}
\]

with some initial and boundary conditions where \( \omega \) is the porosity, \( u \) is the concentration of the dissolved chemical solute carried by the fluid, \( p \) is the pressure, \( v \) is the velocity, further, \( \alpha, k, b \) are given constants, \( K \) and \( g \) are given real functions. For the details of making this model and on flow in such media see \([9, 12]\) and the references there. In \([4, 6]\) a generalization of the above system was studied by using the theory of operators of monotone type. Existence of weak solutions, boundedness and stabilization of solutions were proved. In what follows we investigate a generalization of this model where also the main parts may contain functional dependence on the unknown functions. We show existence and some properties of weak solutions by combining the results and methods of the above mentioned papers. Finally, some examples are given.

1.1 Notation

Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with the uniform \( C^1 \) regularity property (see \([1]\)), further, let \( 0 < T < \infty \), \( 2 \leq p_1, p_2 < \infty \) be real numbers. In the following, \( Q_T := (0,T) \times \Omega \), \( Q_\infty := (0,\infty) \times \Omega \). Denote by \( W^{1,p_i}(\Omega) \) the usual Sobolev space with the norm

\[
\|v\|_{W^{1,p_i}(\Omega)} = \left( \int_\Omega \left(|v|^{p_i} + \sum_{j=1}^n |D_jv|^{p_i} \right) \right)^{1/p_i}
\]

where \( D_j \) denotes the distributional derivative with respect to the \( j \)-th variable (later we use the notation \( D = (D_1,\ldots,D_n) \)). In addition, let \( V_i \) be a closed linear subspace of the space \( W^{1,p_i}(\Omega) \) which contains \( W_0^{1,p_i}(\Omega) \) (the closure of \( C_0^\infty(\Omega) \) in \( W^{1,p_i}(\Omega) \)), and let \( L^{p_i}(0,T; V_i) \) be the Banach space of measurable functions \( u: (0,T) \to V_i \) such that \( \|u\|^{p_i}_{V_i} \) is integrable and the norm is given by

\[
\|u\|_{L^{p_i}(0,T; V_i)} = \left( \int_0^T \|u(t)\|^{p_i}_{V_i} \right)^{1/p_i}.
\]

*This work was supported by the Hungarian National Foundation for Scientific Research under grant OTKA T 049819. This paper is in final form and no version of it is submitted for publication elsewhere.

2000 Mathematics Subject Classification: 35K60, 35J60

Key words and phrases: flow in porous medium, system of nonlinear partial differential equations, functional differential equation
The dual space of $L^p(0,T;V_i)$ is $L^q(0,T;V_i^*)$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $V_i^*$ is the dual of $V_i$. In what follows, we use the notation $X_i := L^p(0,T;V_i)$. The pairing between $V_i^*$, $V_i$ and $X_i^*$, $X_i$ is denoted by $(\cdot, \cdot)$ and $[\cdot, \cdot]$ respectively, further, $D_iu$ stands for the derivative (with respect to the variable $t$) of a function $u \in L^p(0,T;V_i)$. It is well known (see [15]) that if $u \in X_i$, $D_iu \in X_i^*$ then $u \in C([0,T],L^2(\Omega))$ so that $u(0)$ makes sense.

1.2 Formulation of the problem

Let us consider the following system of equations:

\begin{align}
D_0\omega(t,x) &= f(t, x, \omega(t,x), u(t,x); u), \quad \omega(0,x) = \omega_0(x), \\
D_1u(t,x) &= \sum_{i=1}^{n} D_1[a_i(t,x,\omega(t,x), u(t,x), Du(t,x), p(t,x), Dp(t,x); \omega, u, p)] + a_0(t,x,\omega(t,x), u(t,x), Du(t,x), p(t,x), Dp(t,x); \omega, u, p) = g(t,x), \quad u(0,x) = 0, \\
- \sum_{i=1}^{n} D_1[b_i(t,x,\omega(t,x), u(t,x), p(t,x), Dp(t,x); \omega, u, p)] + b_0(t,x,\omega(t,x), u(t,x), p(t,x), Dp(t,x); \omega, u, p) = h(t,x)
\end{align}

with some boundary conditions. This system is a generalization of the model (1)-(4), functions $f, a, b$ may contain nonlocal dependence on the unknown functions $\omega, u, p$ which are written after the symbol "." In the next section we formulate some assumptions on these functions then we may define the weak form of the above system and prove existence of weak solutions.

1.3 Assumptions

In what follows, $\xi_i (\zeta_0, \zeta), (\eta_0, \eta)$ refer for the variables $\omega_i (u, Du)$ and $(p, Dp)$ respectively, further, $w, v_1$ and $v_2$ for the nonlocal dependence on $\omega, u$ and $p$.

A1. For fixed $(w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2$ functions $a_i : Q_T \times \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times L^\infty(Q_T) \times X_1 \times X_2 \to \mathbb{R}$ ($i = 0, \ldots, n$) have the Carathéodory property, i.e., they are measurable in $(t, x) \in Q_T$ for every $(\xi_0, \zeta_0, \zeta, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ and continuous in $(\xi_0, \zeta_0, \zeta, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ for a.a. $(t, x) \in Q_T$.

A2. There exist a constant $c_1 : \mathbb{R} \to \mathbb{R}^+$ and bounded operators $c_1 : L^\infty(Q_T) \times X_1 \times X_2 \to \mathbb{R}^+$, $k_1 : L^\infty(Q_T) \times X_1 \times X_2 \to L^p(Q_T)$ such that

\begin{align}
|a_i(t,x,\xi_0,\zeta,\eta_0,\eta; w, v_1, v_2)| &\leq c_1(w, v_1, v_2)c_1(\xi_0) \left( |\xi_0|^{p_1 - 1} + \left| \frac{\xi_0}{|\eta_0|} \right|^{p_2} + |\eta_0|^{p_2} + |k_1(w, v_1, v_2)|(t,x) \right),
\end{align}

for a.a. $(t, x) \in Q_T$, every $(\xi_0, \zeta_0, \zeta, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ and $(w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2$ ($i = 0, \ldots, n$).

A3. There exists a constant $C > 0$ such that for a.a. $(t, x) \in Q_T$, every $(\xi_0, \zeta_0, \zeta, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ and $(w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2$

\begin{align}
\sum_{i=1}^{n} \left( a_i(t,x,\xi_0,\zeta,\eta_0,\eta; w, v_1, v_2) - a_i(t,x,\xi_0,\tilde{\zeta},\eta_0,\eta; w, v_1, v_2) \right) (\zeta_i - \tilde{\zeta}_i) \geq C |\zeta - \tilde{\zeta}|^{p_1}.
\end{align}

A4. There exist constants $c_2 > 0$, a continuous function $\gamma : \mathbb{R} \to \mathbb{R}$ and bounded operators $\Gamma : L^\infty(Q_T) \to L^\infty(Q_T)$, $k_2 : X_1 \to L^1(Q_T)$ such that

\begin{align}
\sum_{i=0}^{n} a_i(t,x,\xi_0,\zeta_0,\eta_0,\eta; w, v_1, v_2) \zeta_i \geq c_2 (|\xi_0|^{p_1} + |\zeta_0|^{p_1} - \gamma(\xi_0)\Gamma(w)(t,x)|k_2(v_1)|(t,x)
\end{align}

for a.a. $(t, x) \in Q_T$ and every $(\xi_0, \zeta_0, \zeta, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, $(w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2$. Further,

\begin{align}
\lim_{\|v_1\|_{X_1} \to +\infty} \frac{\|k_2(v_1)\|_{L^1(Q_T)}}{|v_1|^{\alpha}_{X_1}} = 0.
\end{align}

A5. If $(\omega_k)$ is bounded in $L^\infty(Q_T)$, $\omega_k \to \omega$ a.e. in $Q_T$ and $u_k \to u$ weakly in $X_1$, strongly in $L^{p_1}(Q_T)$, further, $p_k \to p$ strongly in $X_2$ then

\begin{align}
\lim_{k \to \infty} \|a_i(\cdot, \omega_k, u_k, Du_k, p_k, Dp_k; \omega_k, u_k, p_k) - a_i(\cdot, \omega, u_k, Du_k, p_k, Dp_k; \omega, u, p)\|_{L^{p_1}(Q_T)} = 0.
\end{align}
B1. For fixed \((w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2\) functions \(b_i: Q_T \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1} \times L^\infty(Q_T) \times X_1 \times X_2 \to \mathbb{R}\) have the Carathéodory property, i.e., they are measurable in \((t, x) \in Q_T\) for every \((\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}\) and continuous in \((\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}\) for a.a. \((t, x) \in Q_T\).

B2. There exist a constant \(\hat{c}_1: \mathbb{R} \to \mathbb{R}^+\) and bounded operators \(\hat{c}_1: L^\infty(Q_T) \times X_1 \times X_2 \to \mathbb{R}^+, \quad \hat{k}_1: L^\infty(Q_T) \times X_1 \times X_2 \to L^q_\infty(Q_T)\) such that

\[
|b_i(t, x, \xi, \zeta_0, \eta_0, \eta; w, v_1, v_2)| \leq \hat{c}_1(w, v_1, v_2) \hat{c}_1(\xi) \left(|\eta_0|^{p_2-1} + |\eta|^{p_2-1} + |\zeta_0|^{p_2} + |\hat{k}_1(w, v_1, v_2)|(t, x)\right)
\]

for a.a. \((t, x) \in Q_T\) and every \((\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}, (w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2\) and \((w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2\)

\[
\sum_{i=0}^n \left(b_i(t, x, \xi, \zeta_0, \eta_0, \eta; w, v_1, v_2) - \hat{c}_1(t, x, \xi, \zeta_0, \eta_0, \eta; w, v_1, v_2)\right) (\eta_0 - \hat{v_0}) = \hat{c}_1 \cdot (|\eta_0|^{p_2} + |\eta - \hat{v_0}|^{p_2}).
\]

for a.a. \((t, x) \in Q_T\), and every \((\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}, (w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2\). Further,

\[
\lim_{\|v_2\|_{X_2} \to \infty} \frac{\|\hat{k}_1(v_2)\|}{\|v_2\|} = 0.
\]

(9)

B3. There exists a constant \(\hat{C} > 0\) such that for a.a. \((t, x) \in Q_T\), every \((\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}\) and \((w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2\)

\[
\sum_{i=0}^n \left(b_i(t, x, \xi, \zeta_0, \eta_0, \eta; w, v_1, v_2) - \hat{c}_1(t, x, \xi, \zeta_0, \eta_0, \eta; w, v_1, v_2)\right) (\eta_0 - \hat{v_0}) \geq \hat{C} \cdot (|\eta_0|^{p_2} + |\eta - \hat{v_0}|^{p_2}).
\]

B4. There exist a constant \(\hat{c}_2 > 0\), a continuous function \(\hat{\gamma}: \mathbb{R} \to \mathbb{R}\) and bounded operators \(\hat{\gamma}_1: L^\infty(Q_T) \to L^\infty(Q_T), \quad \hat{k}_2: X_2 \to L^1(Q_T)\) such that

\[
\sum_{i=0}^n \left(b_i(t, x, \xi, \zeta_0, \eta_0, \eta; w, v_1, v_2)\right) (\eta_0 - \hat{v_0}) \geq \hat{c}_2 \cdot (|\eta_0|^{p_2} + |\eta|^{p_2}) - \hat{\gamma}(\xi)[\hat{\gamma}(w)][(t, x)] (|\zeta_0|^{p_1} + |\hat{k}_2(v_2)|(t, x))
\]

for a.a. \((t, x) \in Q_T\), and every \((\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}, (w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2\). Further,

\[
\lim_{\|v_2\|_{X_2} \to \infty} \frac{\|\hat{k}_2(v_2)\|}{\|v_2\|} = 0.
\]

F1. If \((\omega_k)\) is bounded in \(L^\infty(Q_T)\), \(\omega_k \to \omega\) a.e. in \(Q_T\) and \(u_k \to u\) weakly in \(X_1\), strongly in \(L^{p_1}(Q_T)\), further, \(p_k \to p\) weakly in \(X_2\), strongly in \(L^{p_2}(Q_T)\) then

\[
\lim_{k \to \infty} \|b_i(\cdot, \omega_k, u_k, p_k; \omega_k, u_k, p_k) - b_i(\cdot, \omega_k, u_k, p_k; \omega, u, p)\|_{L^{p_2}(Q_T)} = 0.
\]

F2. There is a bounded operator \(\mathcal{B}_2: X_1 \to \mathbb{R}^{+}\) and a continuous function \(\mathcal{B}_2: \mathbb{R} \to \mathbb{R}^{+}\) such that for a.a. \((t, x) \in Q_T\), every \((\xi, \zeta_0) \in \mathbb{R} \times \mathbb{R}^{n+1}\) and \(v \in X_1\)

\[
|f(t, x, \xi, \zeta_0; v) - f(t, x, \xi, \zeta_0; v)| \leq \mathcal{K}_1(v)K_1(\zeta_0) \cdot |\xi - \xi_0|.
\]

F3. There exists \(\omega^* \in L^\infty(\Omega)\) such that for a.a. \((t, x) \in Q_T\), every \((\xi, \zeta_0) \in \mathbb{R} \times \mathbb{R}^{n+1}\) and \(v \in X_1\)

\[
(f(t, x, \xi, \zeta_0; v) - f(t, x, \xi, \zeta_0; v)) = \mathcal{K}_2(\zeta_0) \cdot |\xi - \xi^*| \leq 0.
\]

F4. If \((\omega_k)\) is bounded in \(L^\infty(Q_T)\) and \(u_k \to u\) strongly in \(L^{p_1}(Q_T)\) then

\[
\lim_{k \to \infty} \|f(\cdot, \omega_k, u_k; u_k) - f(\cdot, \omega_k, u_k; u)\|_{L^{p_1}(Q_T)} = 0.
\]
1.4 Weak form

If the above assumptions are satisfied we may define operators $A: L^\infty(Q_T) \times X \times X \to X_1^*$, $B: L^\infty(Q_T) \times X \times X \to X_2^*$ by:

$$[A(\omega, u, p), v_1] := \int_{Q_T} \sum_{i=1}^n a_i(t, x, \omega(t, x), u(t, x), D\omega(t, x), v(t, x), Dp(t, x); \omega, u, p)Dv_1(t, x) \, dt \, dx +$$

$$\quad + \int_{Q_T} a_0(t, x, \omega(t, x), u(t, x), D\omega(t, x), v(t, x), Dp(t, x); \omega, u, p)v_1(t, x) \, dt \, dx,$$  \hspace{1cm} (10)

$$[B(\omega, u, p), v_2] := \int_{Q_T} \sum_{i=1}^n b_i(t, x, \omega(t, x), u(t, x), p(t, x), D\omega(t, x); \omega, u, p)Dv_2(t, x) \, dt \, dx +$$

$$\quad + \int_{Q_T} b_0(t, x, \omega(t, x), u(t, x), p(t, x), D\omega(t, x); \omega, u, p)v_2(t, x) \, dt \, dx,$$ \hspace{1cm} (11)

for $v_1 \in X_1$ and $v_2 \in X_2$. In addition, let us introduce the linear operator $L : D(L) \to X_1^*$ by the formula

$$D(L) = \{ u \in X_1 : D_t u \in X_1^*, u(0) = 0 \}, \quad Lu = D_t u.$$ \hspace{1cm} (12)

By the operators above we may define the weak form of system (5)-(7) as

$$\omega(t, x) = \omega_0(x) + \int_0^t f(s, x, \omega(s, x), u(s, x); u) \, ds \quad \text{for a.a.} \quad (t, x) \in Q_T$$ \hspace{1cm} (13)

$$Lu + A(\omega, u, p) = G$$ \hspace{1cm} (14)

$$B(\omega, u, p) = H$$ \hspace{1cm} (15)

where $G \in X_1^*$ and $H \in X_2^*$ are define by

$$[G, v_1] = \int_{Q_T} g(t, x)v_1(t, x) \, dt \, dx, \quad [H, v_2] = \int_{Q_T} h(t, x)v_2(t, x) \, dt \, dx$$

where $v_i \in X_i$ $(i = 1, 2)$.

2 Existence of solutions

In this section we prove

**Theorem 1.** Suppose that conditions A1–A5, B1–B5, F1–F4 are fulfilled. Then for every $\omega_0 \in L^\infty(\Omega)$, $G \in X_1^*$ and $H \in X_2^*$ there exists a solution $\omega \in L^\infty(Q_T)$, $u \in D(L)$, $p \in L^p(0,T; V_2)$ of problem (13)–(15).

First we formulate some statements related to the solvability of the above equations (13)–(15).

**Proposition 2.** Assume that conditions F1, F3 are satisfied. Then for every fixed $u \in L^p(Q_T)$ and $\omega_0 \in L^\infty(Q_T)$ there exists a unique solution $\omega \in L^\infty(Q_T)$ of the integral equation (13), further, for the solution $\omega$, estimate $\|\omega\|_{L^\infty(Q_T)} \leq \|\omega_0\|_{L^\infty(\Omega)} + \|\omega^x\|_{L^\infty(\Omega)}$ holds.

**Proof.** Immediately follows from Proposition 2.3 in [4] since for fixed nonlocal variable $u$, condition F1 is the same as in the cited paper.

**Proposition 3.** Assume F1–F4 and let $(u_k) \subset L^p(Q_T)$, further, let $\omega_k$ be the solution of (13) corresponding to $u_k$. If $u_k \to u$ in $L^p(Q_T)$ then $\omega_k \to \omega$ a.e. in $Q_T$ where $\omega$ is the solution of (13) corresponding to $u$.

**Proof.** Suppose that $u_k \to u$ in $L^p(Q_T)$ and let $\omega_k, \omega \in L^\infty(Q_T)$ be the corresponding solutions of (13). Then for a.a. $x \in \Omega$, $u_k(\cdot, x) \to u(\cdot, x)$. Fix such a point $x \in \Omega$. Since $(\omega_k)$ is bounded in $L^\infty(Q_T)$ by Proposition 2, we can apply condition F1, F2 and we obtain

$$|\omega_k(t, x) - \omega(t, x)| \leq \int_0^t |f(s, x, \omega_k(s, x), u_k(s, x); u_k) - f(s, x, \omega_k(s, x), u_k(s, x); u)| \, ds +$$

$$\quad + \int_0^t |f(s, x, \omega_k(s, x), u_k(s, x); u) - f(s, x, \omega(s, x), u(s, x); u)| \, ds.$$
The first integral converges to 0 for a.a. \( x \in \Omega \) by condition F4, further, by F1, F2 it is easy to show that the second integral is less then

\[
\text{const} \cdot \left( \int_0^t |\omega_k(s,x) - \omega(s,x)|^{p_2} \, ds \right)^{1/p_2} + \text{const} \cdot \int_0^T |u_k(s,x) - u(s,x)| \, ds.
\]

Hence

\[
|\omega_k(t,x) - \omega(t,x)|^{p_2} \leq \text{const} \cdot \int_0^t |\omega_k(s,x) - \omega(s,x)|^{p_2} \, ds + r(u_k)
\]

where the remainder term \( r(u_k) \) tends to 0 as \( k \to \infty \). Thus Gronwall’s lemma yields \( |\omega_k(t,x) - \omega(t,x)| \leq \text{const} \cdot r(u_k) \to 0 \) which implies the desired a.e. convergence of \( \omega_k \).

\[\square\]

**Proposition 4.** Assume A1–A5. Then for every fixed \( \omega \in L^\infty(Q_T) \), \( p \in X_2 \) and \( G \in X_1^* \) there exists a solution \( u \in D(L) \) of problem \( Lu + A(\omega,u,p) = G \).

**Proof.** The proof follows from Theorem 1.1 in [13] (based on the theory of monotone type operators, see [2]) since for fixed \( \omega \in L^\infty(Q_T) \) and \( p \in X_2 \) conditions A1–A5 imply that operator \( A(\omega_v,p) \): \( X_1 \to X_2^* \) fulfills conditions I–V of the mentioned theorem.

\[\square\]

**Proposition 5.** Suppose that B1–B5 hold. Then for every fixed \( \omega \in L^\infty(Q_T) \), \( u \in X_1 \) and \( H \in X_2^* \) there exists a solution \( p \in X_2 \) of problem \( B(\omega,u,p) = H \).

**Proof.** The statement follows from the theory of monotone operators (see [15]) since conditions B1–B4 imply the boundedness, demicontinuity, uniform monotonicity and coerciveness of operator \( B(\omega,u,:) \): \( X_2 \to X_2^* \) for fixed \( \omega \in L^\infty(Q_T) \), \( u \in X_1 \).

\[\square\]

**Proof of Theorem 1.** The idea is similar as in [4]. We define sequences of approximate solutions of problem (13)–(15) and we show the boundedness of these sequences. After choosing weakly convergent subsequences we verify that the weak limits of the subsequences are solutions of the problem. For simplicity, in the proof we omit the variable \( (t,x) \) of functions \( a_n, b_n \) if it is not confusing.

**Step 1: approximation.** Define the sequences \( (\omega_k), (u_k), (p_k) \) as follows. Let \( \omega_0(t,x) = u_0(t,x) = p_0(t,x) = 0 \) for \( (t,x) \in Q_T \) and for \( k = 0, 1, \ldots \) let \( \omega_{k+1}, u_{k+1}, p_{k+1} \) be a solution of the system:

\[
\omega_{k+1}(t,x) = \omega_k(t,x) + \int_0^t f(s,x,\omega_{k+1}(s,x),u_k(s,x);u_k) \, ds \quad (16)
\]

\[
Lu_{k+1} + A(\omega_k,u_{k+1},p_k) = G \quad \text{(17)}
\]

\[
B(\omega_k,u_{k+1},p_{k+1}) = H. \hspace{1cm} (18)
\]

By Propositions 2, 4, 5 we have solutions \( \omega_{k+1} \in L^\infty(Q_T) \), \( u_{k+1} \in X_1 \), \( p_{k+1} \in X_2 \) so the above recurrence yields the sequences \( (\omega_k) \subset L^\infty(Q_T), (u_k) \subset X_1, (p_k) \subset X_2 \).

**Step 2: boundedness.** We show that the above defined sequences are bounded. By Proposition 2 for fixed \( \omega_0 \in L^\infty(\Omega) \) the solution of equation (16) estimate \( ||\omega_{k+1}||_{L^\infty(Q_T)} \leq ||\omega_0||_{L^\infty(\Omega)} + ||\omega^*||_{L^\infty(\Omega)} \) holds thus \( (\omega_k) \) is bounded in \( L^\infty(Q_T) \).

Now by choosing the test function \( v = u_{k+1} \) in (17) and by using condition A4 and the monotonicity of operator \( L \) we obtain

\[
[G, u_{k+1}] = [Lu_{k+1}, u_{k+1}] + [A(\omega_k, u_{k+1}, p_k), u_{k+1}] \geq c_2 \int_{Q_T} (||u_{k+1}||_{p_1} + ||Du_{k+1}||_{p_1} - \gamma(\omega_k)k_{1}(u_{k+1})) \geq c_2 \int_{Q_T} ||u_{k+1}||_{X_1}^{p_1-1} \gamma(\omega_k)k_{1}(u_{k+1}) \leq \gamma(\omega_k)||u_{k+1}||_{L^\infty(Q_T)} - \frac{||u_{k+1}||_{L^1(Q_T)}}{1 - K \cdot \frac{||u_{k+1}||_{L^1(Q_T)}}{||u_{k+1}||_{X_1}}},
\]

thus by the boundedness of \( (\omega_k) \) we conclude for some \( K > 0 \)

\[
||u_{k+1}||_{X_1}^{p_1-1} \left( 1 - K \cdot \frac{||u_{k+1}||_{L^1(Q_T)}}{||u_{k+1}||_{X_1}} \right) \leq \text{const}
\]

Now (8) implies that \( (u_k) \) is bounded in \( X_1 \).

The boundedness of \( (p_k) \) in \( X_2 \) follows by similar arguments as above by using condition B4 and the boundedness of the sequences \( (\omega_k), (u_k) \).

We need also the boundedness of the sequence \( (Lu_k) \in X_2^* \). By Hölder’s inequality

\[
||A(\omega_k, u_{k+1}, p_k), v ||_1 \leq \left( \sum_{i=0}^{n} ||a_i(\omega_k, u_{k+1}, Du_{k+1}, p_k; \omega_k, u_{k+1}, p_k)||_{L^{p_1}(Q_T)} \right) \cdot ||v||_{X_1}.
\]
and from condition A2 it follows that for all \( i \)
\[
\|a_i(\omega, u_{k+1}, D u_{k+1}, p_k, D p_k; \omega, u_{k+1}, p_k)\|_{L^2(Q_T)} \leq \text{const} \cdot c_i(\omega) c_1(\omega, u_{k+1}, p_k) \left( \|u_{k+1}\|_{X_1}^2 + \|p_k\|_{X_2}^2 + \|k_1(\omega, u_{k+1}, p_k)\|_{L^2(Q_T)} \right).
\]

Therefore by the boundedness of the sequences \((\omega_k, (u_k), (p_k))\) and the boundedness of operators \(c_1, c_1, k_2\) we conclude \(\|L u_{k+1}\| = \|A(\omega, u_{k+1}, p_k) + G, v\| \leq \text{const} \cdot \|v\|_{X_1}\) so \((L u_k)\) is a bounded sequence in \(X^*_T\).

**Step 3: convergence.** Due to the boundedness of the sequences \((u_k), (L u_k), (p_k)\) (in reflexive Banach spaces) each has a weakly convergent subsequence, further, by applying a well known embedding theorem (see [11]) it follows that there exist subsequences (which will be denoted as the original sequences) and functions \(\omega \in L^\infty(Q_T), u \in X_1, p \in X_2\) such that
\[
\begin{align*}
    u_k & \to u \text{ weakly in } X_1, \text{ strongly in } L^p(Q_T), \text{ a.e. in } Q_T; \\
    L u_k & \to Lu \text{ weakly in } X^*_1; \\
    p_k & \to p \text{ weakly in } X_2.
\end{align*}
\]

In what follows, we show that \(\omega, u, p\) are solutions of problem (13)-(15).

Since \(u_k \to u \in L^p(Q_T)\), further, \(u_{k+1}\) is the solution of equation (16), by Proposition 3 it follows that \(\omega_k \to \omega\) a.e. in \(Q_T\) and functions \(\omega, u\) satisfy the integral equation (13).

Now let us consider equation (18). First we show that \(p_k \to p\) in \(X_2\). To this end, let us introduce operator \(\tilde{B}: L^\infty(Q_T) \times X_1 \times X_2 \times L^\infty(Q_T) \times X_1 \times X_2 \to X^*_2\) by

\[
[\tilde{B}(\omega, u, p; w, v_1, v_2, z)] = \int_{Q_T} \sum_{i=1}^n b_i(t, x, \omega(t, x), u(t, x), p(t, x), D p(t, x); w, v_1, v_2) D z_i(t, x) \, dt \, dx + \int_{Q_T} b_0(t, x, \omega(t, x), u(t, x), p(t, x), D p(t, x); w, v_1, v_2) z_2(t, x) \, dt \, dx
\]

for \(z_2 \in X_2\). Observe \(B(\omega, u, p) = \tilde{B}(\omega, u, p; \omega, u, p)\). By condition B3 we have

\[
[\tilde{B}(\omega_k, u_k, p_{k+1}; \omega, u, p) - \tilde{B}(\omega_k, u_k, p; \omega, u, p), p_{k+1} - p] \geq C \cdot \|p_{k+1} - p\|_{X_2}^2.
\]

On the left hand side of the above inequality we have the following decomposition:

\[
[\tilde{B}(\omega_k, u_k, p_{k+1}; \omega, u, p) - \tilde{B}(\omega_k, u_k, p; \omega, u, p), p_{k+1} - p] = [\tilde{B}(\omega_k, u_k, p_{k+1}; \omega, u_k, p_{k+1}), p_{k+1} - p] + [\tilde{B}(\omega_k, u_k, p_{k+1}; \omega, u, p) - \tilde{B}(\omega_k, u_k, p_{k+1}; \omega, u_k, p_{k+1}), p_{k+1} - p] + [\tilde{B}(\omega_k, u_k, p; \omega, u, p) - \tilde{B}(\omega_k, u_k, p; \omega, u, p), p_{k+1} - p - [\tilde{B}(\omega_k, u_k, p; \omega, u, p), p_{k+1} - p].
\]

We show that each term on the right hand side tends to 0. By recurrence (18), \(\tilde{B}(\omega_k, u_k, p_{k+1}; \omega, u_k, p_{k+1}) = H\), further, \(p_{k+1} \to p\) weakly in \(X_2\) which implies the convergence of the first and the last term. The convergence of the second term follows from condition B3. In order to verify the convergence of the third term, observe that

\[
\|\tilde{B}(\omega_k, u_k, p; \omega, u, p) - \tilde{B}(\omega, u, p; \omega, u, p), p_{k+1} - p\| \leq \sum_{i=0}^n \|b_i(\omega_k, u_k, p, D p; \omega, u, p) - b_i(\omega, u, p, D p; \omega, u, p)\|_{L^q(Q_T)} \cdot \|p_{k+1} - p\|_{X_2}
\]

and by condition B2

\[
\|b_i(\omega_k, u_k, p, D p; \omega, u, p) - b_i(\omega, u, p, D p; \omega, u, p)\|^2 \leq \text{const} \cdot c_i(\omega, u, p) \cdot (|c_1(\omega)|^2 + |c_1(\omega)|^2) \left( |p|^2 + |D p|^2 + |u|^2 + |u|^2 + |b_i(\omega, u, p)|^2 \right).
\]

Due to the boundedness of \((\omega_k)\) in \(L^\infty(Q_T)\) and the convergence of \((u_k)\) in \(L^p(Q_T)\) the left hand side of the above inequality is equi-integrable (see [8]), in addition, it a.e. converges to 0, therefore by Vitali’s theorem the left hand side converges in \(L^1(Q_T)\) to the zero function. Thus (because of the boundedness of \((p_k)\)) the right hand side of (22) tends to 0. Hence all terms on the right hand side of equation (21) converges to 0 thus (20) implies \(p_{k+1} \to p\) in \(X_2\).

Now by using the same arguments as in [4] one obtains that \(\tilde{B}(\omega_k, u_k, p_{k+1}; \omega, u, p) \to \tilde{B}(\omega, u, p; \omega, u, p) = B(\omega, u, p)\) weakly in \(X^*_2\). Further, by condition B5 it is not difficult to see that \(\tilde{B}(\omega_k, u_k, p_{k+1}; \omega, u, p) - B(\omega_k, u_k, p_{k+1}; \omega, u_k, p_{k+1}) \to 0\) strongly in \(X^*_2\) thus \(\tilde{B}(\omega_k, u_k, p_{k+1}; \omega, u_k, p_{k+1}) \to B(\omega, u, p)\). Then from recurrence (18) we conclude \(B(\omega, u, p) = H\), i.e., \(\omega, u, p\) are solutions of problem (15).

Finally, \(A(\omega, u, p) = G\) can be shown by similar arguments as above. The proof of the theorem is complete.
3 Examples

We show some examples for functions satisfying conditions A1-A5, B1-B5. Let functions $a_i, b_i$ have the form

$$a_i(t, x, ξ, ζ_0; η, η; w, v_1, v_2) = |ρ_0(t)|\frac{|φ(v_1)(t)|}{|v_1|^\beta}.$$

where $i \neq 0$.\(^{(23)}\)

$$a_0(t, x, ξ, ζ_0; η, η; w, v_1, v_2) = |ρ_0(t)|\frac{|φ(v_1)(t)|}{|v_1|^\beta}.$$

where $i \neq 0$.\(^{(24)}\)

$$b_i(t, x, ξ, ζ_0; η, η; w, v_1, v_2) = |ρ_0(t)|\frac{|φ(v_1)(t)|}{|v_1|^\beta}.$$

where $i \neq 0$.\(^{(25)}\)

where $1 \leq r_i < p_i - 1 (i = 1, 2)$ and the following hold.

E1. a) Operators $π: L^∞(Q_T) \to L^∞(Q_T), φ: L^p_1(Q_T) \to L^∞(Q_T), ψ: X_2 \to L^∞(Q_T)$ are bounded, $φ$ and $ψ$ are continuous, further, if $(ω_k)$ is bounded in $L^∞(Q_T)$ and $ω_k \to ω$ a.e. in $Q_T$ then $π(ω_k) \to π(ω)$ in $L^∞(Q_T)$. In addition, $P ∈ C(ℝ), Q ∈ C(ℝ^{n+1}) \cap L^∞(ℝ^{n+1})$, and there exists a positive lower bound for the values of $π, φ, ψ, P, Q$.

b) Operators $π, π_0: L^∞(Q_T) \to L^∞(Q_T), φ, φ_0: L^p_1(Q_T) \to L^p_1(1)(Q_T)$ are bounded, $φ$ and $φ_0$ are continuous, further, if $(ω_k)$ is bounded in $L^p_1(Q_T)$ and $ω_k \to ω$ a.e. in $Q_T$ then $π(ω_k) \to π(ω)$ and $π_0(ω) \to π_0(ω)$ in $L^∞(Q_T)$. In addition, $P, P_0 \in C(ℝ), π, φ$ and function $P$ are nonnegative and

$$\lim_{||v_1||_X_2 \to +∞} \int_{Q_T} |φ_0(v_1)| = 0.$$

E2. a) Operators $κ: L^∞(Q_T) \to L^∞(Q_T), λ: L^p_1(Q_T) \to L^∞(Q_T), ς: L^p_2(Q_T) \to L^∞(Q_T)$ are bounded, $λ$ and $ς$ are continuous, further, if $(ω_k)$ is bounded in $L^∞(Q_T)$ and $ω_k \to ω$ a.e. in $Q_T$ then $κ(ω_k) \to κ(ω)$ in $L^∞(Q_T)$. In addition, $R ∈ C(ℝ), S ∈ C(ℝ) \cap L^∞(ℝ)$, and there exists a positive lower bound for the values of $κ, λ, ς, R, S$.

b) Operators $κ, κ_0: L^∞(Q_T) \to L^∞(Q_T)$, $κ, κ_0: L^p_1(Q_T) \to L^p_1(1)(Q_T)$ are bounded, $κ$ is continuous, function $R ∈ C(ℝ)$, further, if $(ω_k)$ is bounded in $L^∞(Q_T)$ and $ω_k \to ω$ a.e. in $Q_T$ then $κ(ω_k) \to κ(ω)$ in $L^∞(Q_T)$. In addition, operators $κ, κ_0$ and function $R ∈ C(ℝ)$ are nonnegative and

$$\lim_{||v_2||_X_2 \to +∞} \frac{1}{||v_1||_X_2} \int_{Q_T} |κ_0(v_2)| = 0.$$


By using Young's and Hölder's inequalities it is not difficult to prove the above statement, a detailed proof can be found in [3].

Operators $π, π_0, κ, κ$ may have the form $|π(t)| = \int_{Q_t} |v|^β$, where $1 \leq β$. Further, operators $φ, λ$ may have one of the forms

$$|φ(v)(t)| = \frac{|v|^β}{|v|^β} \text{ or } \frac{|v|^β}{|v|^β} \Phi \left( \int dQ_t \right),$$

where $1 \leq β \leq p_1, d \in L^p_1(Q_T), Φ ∈ C(ℝ) \text{ and } Φ \geq const > 0$. Similarly, $ψ$ may have in the form

$$|ψ(v)(t)| = \frac{|v|^β}{|v|^β} \Psi \left( \int dQ_t \right),$$

where $1 \leq β \leq p_2, d_1, d_2 \in L^p_2(Q_T), Ψ ∈ C(ℝ) \text{ and } Ψ \geq const > 0$. For $φ$ consider, e.g.,

$$|φ(v)(t)| = \frac{|v|^β}{|v|^β} \Phi \left( \int d(s, x)v(s, x) ds \right), \Phi \left( \int d(t, x)v(t, x) dx \right) \text{ or } \Phi \left( \int d(t, x)v(t, x) dx \right),$$

where $d \in L^∞(Q_T), 1 \leq β \leq p_1, d \in C(ℝ), Φ \geq 0$ and $|φ(v)| \leq \text{const} \cdot |v|^β$. In the case of $φ_0$ one has similar examples as for $φ$ above, except $Φ$ does not have to be nonnegative.
For operators \( \nu, \tilde{\nu} \) we may consider similar examples as for \( \varphi, \tilde{\varphi} \) above, by replacing exponents \( p_1 \) with \( p_2 \) and \( r_1 \) with \( r_2 \).

It is not difficult to show that the above operators fulfill conditions E1–E2, for similar arguments see, e.g., [3].

As an example for function \( f \) consider, e.g., \( f(t, x, \xi, \zeta, v; v) = -[\varphi(v)](t, x) f_1(t, x) f_2(\zeta_0) (\xi - \omega^*(x)) \), where \( \varphi \): \( L_1^p(Q_T) \to L_1^\infty(Q_T) \) is bounded and nonnegative, further, \( f_1 \in L_1^\infty(Q_T) \), \( f_2 : \mathbb{R} \to \mathbb{R} \) is nonnegative, Lipschitz continuous and \( |f_2(\zeta_0)| \leq \text{const} |\zeta_0|^{\frac{p_2}{2}} \).

### 4 Solutions in \((0, \infty)\)

In the previous section we have proved existence of solutions for all finite time interval \((0, T)\). In what follows we shall show existence of weak solutions in \((0, \infty)\). Denote by \( X_1^\infty = L_1^p(0, \infty; V_1) \) the space of measurable functions \( u : (0, T) \to V_1 \) for every \( 0 < T < \infty \), further, let \( L_1^p(Q_\infty) \) be the space of functions \( \omega : Q_\infty \to \mathbb{R} \) such that \( \omega|_{Q_T} \in L_1^p(Q_T) \) for every \( 0 < T < \infty \). In the following we suppose

\[ \text{(Vol)} \]

Functions \( a_i : Q_\infty \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) for \( i = 0, \ldots, n \), \( f : Q_\infty \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) have the Volterra property, i.e., \( a_i(t, x, \xi, \zeta_0, \xi_0, \eta, \omega, v_1, v_2)|_{Q_T} \), \( b_i(t, x, \xi, \zeta_0, \xi_0, \eta, v, \omega, v_1, v_2)|_{Q_T} \), \( f(t, x, \xi, \zeta_0, \xi_0, v_1, v_2)|_{Q_T} \) depend only on \( (\omega|_{Q_T}, v_1|_{Q_T}, v_2|_{Q_T}) \) for every \( 0 < T < \infty \).

Now we may define the weak form of \((5)-(7)\) in \( Q_\infty \). For \( 0 < T < \infty \) introduce operators \( A_T : L_1^p(Q_T) \times L_1^p(0, T; V_1) \times L_1^p(0, T; V_2) \to L_1^p(0, T; V_1^2) \), \( B_T : L_1^p(Q_T) \times L_1^p(0, T; V_1) \times L_1^p(0, T; V_2) \to L_1^p(0, T; V_2) \), \( L_T : D(Q_T) \to L_1^p(0, T; V_2^2) \) by formulas \((10)-(12)\). We say that \( \omega \in L_1^p(Q_\infty) \), \( u \in X_1^\infty \), \( p \in X_2^\infty \) is a solution of \((5)-(7)\) in \((0, \infty)\) if for all \( 0 < T < \infty \) (for the restrictions of the functions to \( Q_T \))

\[
\omega(t, x) = \omega(0, x) + \int_0^t f(s, x, \omega(s, x), u(s, x); u) \, ds \quad (t, x) \in Q_T
\]

\[
L_T u + A_T(\omega, u, p) = G_T
\]

\[
B_T(\omega, u, p) = H_T
\]

where \( G_T = G|_{(0, T)} \), \( H_T = H|_{(0, T)} \) with \( G \in L_1^p(0, \infty; V_1^2) \), \( H \in L_1^p(0, \infty; V_2^2) \). Observe that the Volterra property ensures that if \( \omega, u, p \) is a solution in \((0, T)\) for some \( T \) then these functions are solutions in \((0, \tilde{T})\) for all \( \tilde{T} < T \).

**Theorem 7.** Suppose that \((\text{Vol}), A1-A5, B1-B5, F1-F4\) hold (in the sense that they are satisfied by the restrictions of functions \( a_i, b_i, f \) to \( Q_T \) for all \( 0 < T < \infty \)). Then there exists \( \omega \in L_1^\infty(Q_\infty), u \in X_1^\infty, p \in X_2^\infty \) such that \( \omega|_{Q_T}, u|_{Q_T}, p|_{Q_T} \) is a solution of problem \((13)-(15)\) for all \( 0 < T < \infty \).

**Idea of the proof.** One may apply the arguments of the proof of Theorem 1 in [4] word for word. The idea is the following. Due to Theorem 1 we have solutions in \((0, T_0)\) where \( T_0 \to \infty \). By showing the boundedness of these solutions and using a diagonal process one may choose weakly convergent subsequences of solutions. After taking \( k \to \infty \) we obtain a solution in \((0, \infty)\).

**Remark 8.** Examples \((23)-(25)\) fulfill the conditions of the above theorem if operators \( \pi, \tilde{\pi}, \pi_0, k, \tilde{k} : L_1^p(Q_\infty) \to L_1^\infty(Q_\infty), \varphi, \psi, \psi : L_1^p(Q_\infty) \to L_1^\infty(Q_\infty), \varphi, \tilde{\varphi}, \tilde{\varphi} : L_1^p(Q_\infty) \to L_1^\infty(Q_\infty), \tilde{\varphi} : L_1^p(Q_\infty) \to L_1^\infty(Q_\infty), \tilde{\varphi} : L_1^p(Q_\infty) \to L_1^\infty(Q_\infty) \) are of Volterra type and conditions E1–E2 are satisfied for all finite time \( > 0 \). E.g., the operators given after Proposition 6 serve as an example for the above.

### 4.1 Boundedness

Now we show that under some further assumptions, the solutions, formulated in the previous theorem are bounded in appropriate norms in the time interval \((0, \infty)\). First suppose

A4*. There exist a constant \( c_2 > 0 \), a continuous function \( \gamma : \mathbb{R} \to \mathbb{R} \) and Volterra operators \( \Gamma : L_1^\infty(Q_\infty) \to L_1^\infty(Q_\infty) \), \( k_2 : X_1^\infty \to L_1^1(Q_\infty) \) such that

\[
\sum_{i=0}^n a_i(t, x, \xi, \zeta, \eta, v, v_1, v_2) \zeta_i \geq c_2 (|\zeta_0|^p + |\zeta|^p) - \gamma(\xi)[\Gamma(w)](t, x)|k_2(v_1)|(t, x)
\]
for a.a. \((t, x) \in Q_\infty\), every \((\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}\) and \((w, v_1, v_2) \in L^\infty(Q_\infty) \times X_1^\infty \times X_2^\infty\). Further, (8) holds and for all \(v \in X_2^\infty\) uniformly in \(T\)

\[
\int_\Omega ||k_2(v_1)|(t, x)|\, dx \leq \alpha_1 \sup_{\tau \in [0, t]} \|v_1(\tau)\|_{L^2(\Omega)}^{p_1} + \chi_1(t) \sup_{\tau \in [0, t]} \|v_1(\tau)\|_{L^2(\Omega)}^{p_1} + 1
\]

with some \(\alpha_1 > 0, p_1 < p_1\) and \(\chi_1 : \mathbb{R} \to \mathbb{R}\) such that \(\lim_{t \to \infty} \chi_1(t) = 0\).

**B4**. There exist a constant \(\hat{c_2} > 0\), a continuous function \(\hat{\gamma} : \mathbb{R} \to \mathbb{R}\) and Volterra operators \(\hat{\Gamma} : L^\infty(Q_\infty) \to L^\infty(Q_\infty)\), \(k_2 : X_2^\infty \to L^2_{\text{loc}}(Q_\infty)\) such that

\[
\sum_{i=0}^n b_i(t, x, \xi, \zeta_0, \eta_0, \eta, w, v_1, v_2) \eta_i \geq \hat{c_2} (|\eta_0|^{p_2} + |\eta|^{p_2}) - \hat{\gamma}(\xi)(\hat{\Gamma}(w))(t, x) \left(|\zeta_0|^2 + |k_2(v_2)|(t, x)\right)
\]

for a.a. \((t, x) \in Q_\infty\), every \((\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}\) and \((w, v_1, v_2) \in L^\infty(Q_\infty) \times X_1^\infty \times X_2^\infty\). Further, (9) holds and for all \(v \in X_2^\infty\) uniformly in \(T\)

\[
\int_\Omega ||k_2(v_2)|(t, x)|\, dx \leq \alpha_2 \left[ \sup_{\tau \in [0, t]} \|v_2(\tau)\|_{L^2(\Omega)}^{p_2} + \chi_2(t) \sup_{\tau \in [0, t]} \|v_2(\tau)\|_{L^2(\Omega)}^{p_2} + 1 \right]
\]

with some \(\alpha_2 > 0, p_2 < p_2\) and \(\chi_2 : \mathbb{R} \to \mathbb{R}\) such that \(\lim_{t \to \infty} \chi_2(t) = 0\).

**Theorem 9.** Suppose \(p_1, p_2 > 2\) and conditions (Vol), A1–A3, A4*, A5, B1–B3, B4*, B5, F1–F4 are satisfied, further, \(||G(\cdot)||_{L^\infty}, ||H(\cdot)||_{L^\infty} \in L^\infty(0, \infty)\). Then for the solutions \(\omega, u, p\) formulated in Theorem 9, \(\omega \in L^\infty(Q_\infty), u \in L^\infty(0, \infty; L^2(\Omega)), p \in L^\infty(0, \infty; V_2)\).

**Idea of the proof.** We may apply the arguments of the proof of Theorem 2 in [6]. Introduce the notation \(y(t) = |u(t)|_{L^2(\Omega)}^{p_1}\) (then \(y\) is continuous see, e.g. [15]). By choosing arbitrary \(0 < T_1 < T_2 < \infty\), (13), conditions A4*, G2, Young’s inequality and the continuous embedding \(V_1 \hookrightarrow L^2(\Omega)\) imply

\[
\frac{1}{2} (y(T_1) - y(T_2)) + \frac{1}{2} \int_{T_1}^{T_2} y(t)^{p_2} \, dt \leq \operatorname{const} \cdot \int_{T_1}^{T_2} \left[ \sup_{\tau \in [0, t]} y(\tau)^{p_2} + \chi_1(t) \sup_{\tau \in [0, t]} y(\tau)^{p_2} + 1 \right] \, dt.
\]

It is not difficult to see that the above inequality implies the boundedness of \(y\) in \((0, \infty)\), a detailed argument can be found in [14]. The boundedness of \(p\) follows from the boundedness of \(y\) similarly as above, by using conditions B4*, see [6].

**Remark 10.** Example (23)–(25) fulfill the conditions of Theorem 9 if assumptions of Remark 8 are satisfied, in addition

\[
\int_\Omega ||\hat{\varphi_0}(v_1)|(t, x)|\, dx \leq \alpha_1 \left[ \sup_{\tau \in [0, t]} \|v_1(\tau)\|_{L^2(\Omega)}^{p_1} + \chi_1(t) \sup_{\tau \in [0, t]} \|v_1(\tau)\|_{L^2(\Omega)}^{p_1} + 1 \right]
\]

for all \(v_1 \in L^{p_1}_{\text{loc}}(Q_\infty)\) with some constants \(\alpha_1 > 0, \varrho_1 < p_1\) and function \(\chi_1 : \mathbb{R} \to \mathbb{R}\) such that \(\lim_{t \to \infty} \chi_1(t) = 0\), further, similar condition holds for \(\hat{\varphi}\) (by changing the indexes from 1 to 2, and \(L^2(\Omega) \to V_2\)). For example, operator \(\hat{\varphi}_0\) may have the form

\[
[\hat{\varphi}(v)](t, x) = \hat{\Phi} \left( \int_\Omega |d(t, x)||v(t, x)|^{\beta} \, dx \right), \quad \hat{\Phi} \left( \int_\Omega |d(t, x)||v(t, x)|^{\beta} \, dx \right) \quad \text{or} \quad \chi_1(t) \hat{\Phi}_0 \left( \left[ \int_\Omega |d(t, x)||v(t, x)|^2 \, dx \right]^{\frac{1}{2}} \right),
\]

where \(d \in L^\infty(Q_\infty), 1 \leq \beta \leq 2, \hat{\Phi}, \hat{\Phi}_0, \chi_1 \in C(\mathbb{R})\) and \(|\hat{\Phi}(\tau)| \leq \operatorname{const} \cdot |\tau|^{p_1 - \varrho_1 - 1}, |\hat{\Phi}_0(\tau)| \leq \operatorname{const} \cdot |\tau|^{p_1 - \varrho_1 - 1}, \lim_{\tau \to \infty} \chi_1(\tau) = 0\).

**4.2 Stabilization**

In this section we consider a special case of problem (26)–(28), namely, let \(p_1 = p_2 = p\) (thus \(q_1 = q_2 = q, V_1 = V_2 = V\) and \(X_1 = X_2 = X\)). In what follows, we prove stabilization of the solutions of the system, that is, we show the convergence (in some sense) of solutions as \(t \to \infty\) to the solutions of a stationary system. We need some further assumptions:
A6. There exist Carathéodory functions $a_{i,∞}: Ω × ℝ × ℝ^{n+1} → ℝ$ such that for a.a. $x ∈ Ω$ and every $(ξ, η) ∈ Ω × [0, +∞)$, $a_i(x, ξ, u, η) = a_i(ξ, η)$.

$$\lim_{i→∞} a_i(x, ξ, η) = a_i(x, ξ, η) = a_i(ξ, η).$$

B6. There exist Carathéodory functions $b_i: Ω × ℝ × ℝ^{n+1} → ℝ$ such that for a.a. $x ∈ Ω$ and every $(ξ, η) ∈ Ω × [0, +∞)$, $b_i(x, ξ, u, η) = b_i(ξ, η)$.

$$\lim_{i→∞} b_i(x, ξ, u, η) = b_i(x, ξ, u, η) = b_i(ξ, η).$$

AB There exists a constant $C$ such that for a.a. $(x, ξ, η) ∈ [0, +∞)$, $(ξ, η) ∈ [0, +∞)$, and every $(ξ, η) ∈ [0, +∞)$.

$$\sum_{i=0}^{n} \left( a_i(x, ξ, u, η; v, w, v, w, v, w, v, w, v, w, v, w, v, w, v, w, v, w, v, w) \right) = \left( ξ_i - ξ_i^+ \right) + \sum_{i=0}^{n} \left( b_i(x, ξ, u, η; v, w, v, w, v, w, v, w, v, w, v, w, v, w, v, w, v, w, v, w) \right) = \left( η - η \right).$$

F5. For every fixed $v ∈ L^∞(0, +∞)$ there is a constant $m > 0$ such that $(ξ - ξ^+(x)) f(t, x, ξ; v) ≤ -m(ξ - ξ^+(x))^2$ for a.a. $(t, x) ∈ Ω$ and every $(ξ, ξ^+) ∈ [0, +∞)$.

Now introduce operators $A_0: L^∞(Ω) × V → V^*$, $B_0: L^∞(Ω) × V → V^*$ by

$$\langle A_0(ω, u, p), v \rangle := \int_Ω \sum_{i=1}^{n} a_i(x, ω, u, v, w, v, w, v, w, v, w, v, w, v, w, v, w, v, w, v, w, v, w) D_i v(x) dx + \int_Ω a_0(x, u, v, w, v, w, v, w, v, w, v, w, v, w, v, w, v, w, v, w, v, w) v(x) dx,$$

$$\langle B_0(ω, u, p), v \rangle := \int_Ω \sum_{i=1}^{n} b_i(x, ω, u, v, w, v, w, v, w, v, w, v, w, v, w, v, w, v, w, v, w, v, w) D_i v(x) dx + \int_Ω b_0(x, u, v, w, v, w, v, w, v, w, v, w, v, w, v, w, v, w, v, w, v, w) v(x) dx.$$

Theorem 11. Assume conditions A1-A3, A4*, A5-A6, B1-B3, B4*, B5-B6, AB, F1-F5 are satisfied (with $p = p_1 = p_2$), further, there exist $F_0, G_0 ∈ V^*$ such that

$$\lim_{i→∞} ∥F(t) - F_0∥_{V^*} = 0, \quad \lim_{i→∞} ∥G(t) - G_0∥_{V^*} = 0.$$

Then there exist $u_0, p_0 ∈ V$ such that the solutions $ω, u, p$ of problem (26)-(28) are in $L^∞(Ω)$, $u(t) → u_∞$ in $L^2(Ω)$, $\int_{t-1}^{t+1} ∥u(s) - u_∞∥^p ds → 0, ∫_{t-1}^{t+1} ∥p(s) - p_∞∥^p ds → 0$, further,

$$A_0(ω^*, u_∞, p_∞) = G_∞ \quad (29)$$

$$B_0(ω^*, u_∞, p_∞) = H_∞. \quad (30)$$

Sketch of the proof. We follow the proof of Theorem 3 in [6]. Let $ω, u, p$ be solutions of (26)-(28) then by Theorem 9 $ω ∈ L^∞(Ω)$, $u ∈ L^∞(0, +∞; L^2(Ω))$, $p ∈ L^∞(0, +∞; V_2)$. By using the same arguments as in the above mentioned paper, conditions F3, F5 imply estimate $∥ω(t, ·) - ω^*(·)∥L^∞(Ω) ≤ ∥ω_0∥L^∞(Ω)e^{-mt}$ which yields the convergence $ω(t, ·) → ω^*(·)$ in $L^∞(Ω)$.

Now by the using the idea of Proposition 5 and condition AB it is easy to see that for fixed $ω^*$ there exist a unique solution $u_∞, p_∞ ∈ V$ of problem (29)-(30) see, e.g., [15].

In order to show the desired convergences we prove a differential inequality for $u$ and $p$. From equations (26)-(28) and (29)-(30) we obtain

$$\langle D_t(u(t) - u_∞), u(t) - u_∞ \rangle + ∫_Ω [A(ω, u, p)](t) - A_∞(ω^*, u_∞, p_∞) u(t) - u_∞ + ∫_Ω [B(ω, u, p)](t) - B_∞(ω^*, u_∞, p_∞) p(t) - p_∞ = (G(t) - G_0, u(t) - u_∞) + (F(t) - F_∞, p(t) - p_∞). \quad (31)$$
Observe that the first term equals to \(\frac{1}{2}y'(t)\) where \(y(t) = \int (u(t) - u_\infty)^2\). Further, for the second and third terms of the above equation we have by condition AB and Young’s inequality

\[
\langle [A(\omega, u, p)](t) - A_\infty(\omega^*, u_\infty, p_\infty), u(t) - u_\infty \rangle + \langle [B(\omega, u, p)](t) - B_\infty(\omega^*, u_\infty, p_\infty), p(t) - p_\infty \rangle \geq
\]

\[
\geq C (\|u(t) - u_\infty\|_V^p + \|p(t) - p_\infty\|_V^p) - \frac{\varepsilon p}{p} \|u(t) - u_\infty\|_V^p - \frac{\varepsilon p}{p} \|p(t) - p_\infty\|_V^p - \frac{1}{q \varepsilon} \|\hat{A}(\omega, u_\infty, p_\infty; \omega, u, p)\|_V^q,
\]

\[
- \frac{1}{q \varepsilon} \|\hat{B}(\omega, u_\infty, p_\infty; \omega, u, p)\|_V^q)
\]

for a.a. \(x \in \Omega\) and some \(\varepsilon > 0\) (and operator \(\hat{A}\) is defined similarly as \(\hat{B}\), see (19)). We show that last two terms on the right hand side of the above inequality converges to 0 as \(t \to \infty\). Clearly,

\[
\|\hat{A}(\omega, u_\infty, p_\infty; \omega, u, p)\|_V^q \leq \frac{n}{\varepsilon} \int_0^t \|a_i(t, \omega(t), \cdot, u_\infty, Du_\infty, p_\infty, Dp_\infty; \omega, u, p) - a_i(\omega^*, u_\infty, Du_\infty, p_\infty, Dp_\infty)\|_V^q.
\]

The integrand on the right hand side is a.e. convergent in \(\Omega\) as \(t \to \infty\) by condition A6 and since \(\omega(t, x) \to \omega^*(x)\) for a.a. \(x \in \Omega\). Further, it is integrable in \(\Omega\) by conditions A2, A6 and estimate

\[
\|a_i(t, \omega(t), \cdot, u_\infty, Du_\infty, p_\infty, Dp_\infty) - a_i(\omega^*, u_\infty, Du_\infty, p_\infty, Dp_\infty)\|_V^q \leq \text{const} \cdot \left(\|a_i(\omega)\|_{\mathcal{L}\infty(Q, \omega)} + \|a_i(\omega^*)\|_{\mathcal{L}\infty(Q, \omega^*)}\right) \left(\|u_\infty\|_V^p + \|Du_\infty\|_V^p + \|p_\infty\|_V^p + \|Dp_\infty\|_V^p + \|k_i\|_{\mathcal{L}\infty(\omega)}\right)
\]

thus by Lebesgue’s theorem we obtain

\[
\|\hat{A}(\omega, u_\infty, p_\infty; \omega, u, p)\|_V^q \to 0 \text{ as } t \to \infty.
\]

The convergence of the last term in (32) can be proved similarly.

On the right hand side of (31) by Young’s inequality we obtain

\[
\|G(t) - G_\infty, u(t) - u_\infty\|_V^q + \|F(t) - F_\infty, p(t) - p_\infty\|_V^q \leq \text{const} \cdot \left(\|a_i(\omega)\|_{\mathcal{L}\infty(Q, \omega)} + \|a_i(\omega^*)\|_{\mathcal{L}\infty(Q, \omega^*)}\right) \left(\|u_\infty\|_V^p + \|Du_\infty\|_V^p + \|p_\infty\|_V^p + \|Dp_\infty\|_V^p + \|k_i\|_{\mathcal{L}\infty(\omega)}\right)
\]

where the last two terms tend to 0 as \(t \to \infty\).

Now, by choosing sufficiently small \(\varepsilon\) in (31) and by using (32), (33), the above convergences and the continuous embedding \(L^p(\Omega) \hookrightarrow L^2(\Omega)\) we obtain

\[
y'(t) + \text{const} \cdot y(t)\|\|p(t) - p_\infty\|_V^p \leq \varphi(t)
\]

where \(\varphi(t) \to 0\) as \(t \to \infty\) and the constants are positive. It is not difficult to show that this inequality implies \(\lim_{t \to \infty} y(t) = 0\) (see the proof of Theorem 2 in [14]), furthermore, by integrating (31) over \((t - 1, t + 1)\) one can deduce the convergences

\[
\int_{t - 1}^{t + 1} \|u(s) - u_\infty\|_V^q \, ds \to 0, \quad \int_{t - 1}^{t + 1} \|p(s) - p_\infty\|_V^q \, ds \to 0,
\]

too. The proof of stabilization is complete.

Consider the following functions for \(i = 0, \ldots, n\)

\[
a_i(t, x, \xi, \zeta_0, \zeta, \eta, \eta; w, v_1, v_2) = \left|\pi(w)(t, x)\pi(v_1)(t, x)\pi(v_2)(t, x)P(\xi)\zeta_0(\xi, \eta, \eta)\|p_0\|^{-2}\right|,
\]

\[
b_i(t, x, \xi, \zeta_0, \zeta, \eta, \eta; v_1, v_2) = \left|\pi(w)(t, x)\pi(v_1)(t, x)\pi(v_2)(t, x)R(\xi, \eta)\|p_0\|^{-2}\right|.
\]

Suppose

**E3.**

a) Operators \(\pi: L^\infty_\text{loc}(Q, \omega) \to L^\infty_\text{loc}(Q, \omega), \varphi, \psi: L^\infty_\text{loc}(Q, \omega) \to L^\infty_\text{loc}(Q, \omega)\) are of Volterra type, further, for every \(0 < T < \infty, \pi: L^\infty(Q, \omega) \to L^\infty(Q, \omega), \varphi, \psi: L^\infty(Q, \omega) \to L^\infty(Q, \omega)\) are bounded, \(\varphi, \psi\) are continuous, and if \((\omega_0)\) is bounded in \(L^\infty(Q, \omega)\) and \(w_0 \to \omega\) a.e. in \(Q_T\) then \(\pi(w_0) \to \pi(\omega)\) in \(L^\infty(Q, \omega)\).

b) Suppose \(\pi(\omega) \in L^\infty(\Omega)\) such that \(\pi(\omega_0) \in L^\infty(\Omega)\) and \(\|\pi(\omega_0)\|_{L^\infty(\Omega)} \leq \|\pi(\omega)\|_{L^\infty(\Omega)}\).

\[
\lim_{t \to \infty} \left(\|\pi(w)(t, \cdot) - \pi(\omega)\|_{L^\infty(\Omega)} + \|\varphi(v_1)(t, \cdot) - \varphi(\omega)\|_{L^\infty(\Omega)} + \|\psi(v_2)(t, \cdot) - \psi(\omega)\|_{L^\infty(\Omega)}\right) = 0.
\]
E4. a) Operators $\kappa: L^\infty_{loc}(Q,\infty) \to L^\infty_{loc}(Q,\infty)$, $\lambda, \vartheta: L^p_{loc}(Q,\infty) \to L^\infty_{loc}(Q,\infty)$ are of Volterra type, further, for every $0 < T < \infty$, $\kappa: L^\infty(Q,T) \to L^\infty(Q,T)$, $\lambda, \vartheta: L^p(Q,T) \to L^\infty(Q,T)$ are bounded, $\lambda$ and $\vartheta$ are continuous, and if $(\omega_k)$ is bounded in $L^\infty(Q,T)$ and $\omega_k \to \omega$ a.e. in $Q_T$ then $\kappa(\omega_k) \to \kappa(\omega)$ in $L^\infty(Q_T)$. In addition, $R \in C(\mathbb{R})$, and there exists a positive lower bound for the values of $\pi, \varphi, \psi, R$.

b) There exist $\kappa_\infty, \lambda_\infty, \vartheta_\infty \in L^\infty(\Omega)$ such that $\kappa_\infty, \lambda_\infty, \vartheta_\infty \geq \text{const} > 0$, further, for every $w \in L^\infty(Q,\infty)$, $v_1 \in X^\infty \cap L^\infty(0,\infty;L^2(\Omega)), v_2 \in X^\infty \cap L^\infty(0,\infty;V)$

$$\lim_{t \to \infty} (\|\kappa(w)(t,\cdot) - \kappa_\infty\|_{L^\infty(\Omega)} + \|\lambda(v_1)(t,\cdot) - \lambda_\infty\|_{L^\infty(\Omega)} + \|\vartheta(v_2)(t,\cdot) - \vartheta_\infty\|_{L^\infty(\Omega)}) = 0.$$

It is not difficult to prove (for some arguments see, e.g., [7, 3, 13]).

**Proposition 12.** Suppose $2 \leq p \leq 4$ and E3–E4, then the above (34)–(35) functions satisfy conditions A1–A3, A4*, A5–A6, B1–B3, B4*, B5–B6, AB with $p_1 = p_2 = p$.

If we consider

$$a_i(t,x,\xi,\zeta,\eta,\eta;w,v_1,v_2) = |\xi|(|\xi,\eta|^{p-2} + |\pi(w)(t,x)|\phi(v_1)(t,x)P(\xi)|\xi,\eta|^{r-2},$$

$$b_i(t,x,\xi,\eta,\eta;w,v_1,v_2) = |\eta|(|\eta,\eta|^{p-2} + |\kappa(w)(t,x)|\vartheta(v_2)(t,x)R(\xi)|\xi,\eta|^{r-2}$$

where $1 \leq r \leq 4$ and E3–E4 hold then it is easy to see that these functions satisfy conditions A1–A3, A4*, A5–A6, B1–B6, AB with $p_1 = p_2 = p \geq \max(2,r$. E.g. operators $\pi, \varphi$ may have the form

$$[\pi(w)(t,x) = \chi(t) \int_{Q,T} |w|^{p} + \pi_\infty(x), \varphi(v)(t,x) = \tilde{\chi}(t) \int_{\Omega} |d(t,x)| |v(t,x)|^{p} dx + \varphi_\infty(x),$$

where $\lim_{t \to \infty} \chi(t) = 0, \lim_{t \to \infty} \tilde{\chi}(t) = 0$ and $d, \pi_\infty, \varphi_\infty \in L^\infty(\Omega), 1 \leq \alpha, 1 \leq \beta \leq 2$. The other operators may have similar form.

As an example for function $f$ consider, e.g., $f(t,x,\xi,\zeta;x;v) = -(\xi - \omega^*(x)) \int_{\Omega} |w(t,x)|^{p} dx$ where $1 \leq \beta \leq 2$.

**References**


Ádám Besenyei  
Department of Applied Analysis  
Eötvös Loránd University  
Pázmány Péter sétány 1/C  
H-1117 Budapest  
Hungary  
email: badam@cs.elte.hu