On a nonlinear system containing nonlocal terms related to a fluid flow model

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Abstract

We consider a nonlinear system of differential equations where the main parts may contain nonlocal dependence on the unknowns. This system is a generalization of a model describing fluid flow in porous medium. Existence of weak solutions, boundedness and stabilization of solutions as $t \to \infty$ is shown by using the theory of monotone operators, and some examples are given.

1 Introduction

This paper was motivated by works [3, 4, 6, 13]. In [13] parabolic quasilinear functional differential equations were investigated where also the main part contains functional dependence on the unknown function, for example the integral of it. Such problems may occour, e.g., in a diffusion process for heat or population where the diffusion coefficient depends on nonlocal quantity. In [13] existence of weak solutions in time interval $(0, T)$ $(0 < T \le \infty)$, boundedness and stabilization of solutions as $t \to \infty$ were proved for that type of parabolic problems. Some of these results were extended to systems of equations and to parabolic variational inequalities in [3, 5]. In this paper we extend these results to a system which was motivated by the following one-dimensional model describing fluid flow (carrying chemical species) in porous medium that was studied in [12]:

$$\omega(t,x)u_t(t,x) = \alpha \cdot (|v(t,x)|u_x(t,x))_x + K(\omega(t,x))p_x(t,x)u_x(t,x) - ku(t,x)g(\omega(t,x)) \tag{1}$$

$$\omega_t(t,x) = bu(t,x)g(\omega(t,x)) \tag{2}$$

$$(K(\omega(t,x))p_x(t,x))_x = bu(t,x)g(\omega(t,x)), \tag{3}$$

$$v(t,x) = -K(\omega(t,x))p_x(t,x), \qquad t > 0,\ x \in (0,1), \tag{4}$$

with some initial and boundary conditions where ω is the porosity, u is the concentration of the dissolved chemical solute carried by the fluid, p is the pressure, v is the velocity, further, α, k, b are given constants, K and g are given real functions. For the details of making this model and on flow in such media see [9, 12] and the references there. In [4, 6] a generalization of the above system was studied by using the theory of operators of monotone type. Existence of weak solutions, boundedness and stabilization of solutions were proved. In what follows we investigate a generalization of this model where also the main parts may contain functional dependence on the unknown functions. We show existence and some properties of weak solutions by combining the results and methods of the above mentioned papers. Finally, some examples are given.

1.1 Notation

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the uniform C^1 regularity property (see [1]), further, let $0 < T < \infty$, $2 \le p_1, p_2 < \infty$ be real numbers. In the following, $Q_T := (0,T) \times \Omega$, $Q_\infty := (0,\infty) \times \Omega$. Denote by $W^{1,p_i}(\Omega)$ the usual Sobolev space with the norm

$$\|v\|_{W^{1,p_i}(\Omega)} = \Big(\int_\Omega (|v|^{p_i} + \sum_{j=1}^n |D_j v|^{p_i}) \Big)^{1/p_i}$$

where D_j denotes the distributional derivative with respect to the j-th variable (later we use the notation $D = (D_1, \dots, D_n)$). In addition, let V_i be a closed linear subspace of the space $W^{1,p_i}(\Omega)$ which contains $W_0^{1,p_i}(\Omega)$ (the closure of $C_0^\infty(\Omega)$ in $W^{1,p_i}(\Omega)$), and let $L^{p_i}(0,T;V_i)$ be the Banach space of measurable functions $u\colon (0,T) \to V_i$ such that $\|u\|_{V_i}^{p_i}$ is integrable and the norm is given by

$$\|u\|_{L^{p_i}(0,T;V_i)} = \Big(\int_0^T \|u(t)\|_{V_i}^{p_i} dt \Big)^{1/p_i}.$$

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The dual space of $L^{p_i}(0,T;V_i)$ is $L^{q_i}(0,T;V_i^*)$ where $\frac{1}{p_i}+\frac{1}{q_i}=1$ and V_i^* is the dual of V_i. In what follows, we use the notation $X_i := L^{p_i}(0,T;V_i)$. The pairing between V_i^*, V_i and X_i^*, X_i is denoted by $\langle\cdot,\cdot\rangle$ and $[\cdot,\cdot]$, respectively, further, $D_t u$ stands for the derivative (with respect to the variable t) of a function $u\in L^{p_i}(0,T;V_i)$. It is well known (see [15]) that if $u\in X_i$, $D_t u\in X_i^*$ then $u\in C([0,T],L^2(\Omega))$ so that $u(0)$ makes sense.

1.2 Formulation of the problem

Let us consider the following system of equations:

$$D_t\omega(t,x)=f(t,x,\omega(t,x),u(t,x);u),\qquad \omega(0,x)=\omega_0(x), \tag{5}$$

$$\begin{aligned} D_t u(t,x)-\sum_{i=1}^n D_i\left[a_i(t,x,\omega(t,x),u(t,x),Du(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p})\right]+ \\ +a_0(t,x,\omega(t,x),u(t,x),Du(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p})=g(t,x),\qquad u(0,x)=0,\end{aligned} \tag{6}$$

$$\begin{aligned} -\sum_{i=1}^n D_i[b_i(t,x,\omega(t,x),u(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p})]+ \\ +b_0(t,x,\omega(t,x),u(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p})=h(t,x)\end{aligned} \tag{7}$$

with some boundary conditions. This system is a generalization of the model (1)–(4), functions f,a_i,b_i may contain nonlocal dependence on the unknown functions $\omega,u,\mathbf{p}$ which are written after the symbol ";". In the next section we formulate some assumptions on these functions then we may define the weak form of the above system and prove existence of weak solutions.

1.3 Assumptions

In what follows, ξ, (ζ_0,ζ), (η_0,η) refer for the variables ω, (u,Du) and $(\mathbf{p},D\mathbf{p})$, respectively, further, w, v_1 and v_2 for the nonlocal dependence on ω, u and $\mathbf{p}$.

A1. For fixed $(w,v_1,v_2)\in L^\infty(Q_T)\times X_1\times X_2$ functions $a_i\colon Q_T\times\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}\times L^\infty(Q_T)\times X_1\times X_2\to\mathbb{R}$ $(i=0,\dots,n)$ have the Carathéodory property, i.e., they are measurable in $(t,x)\in Q_T$ for every $(\xi,\zeta_0,\zeta,\eta_0,\eta)\in\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ and continuous in $(\xi,\zeta_0,\zeta,\eta_0,\eta)\in\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ for a.a. $(t,x)\in Q_T$.

A2. There exist a continuous function $c_1\colon\mathbb{R}\to\mathbb{R}^+$ and bounded operators $\mathfrak{c}_1\colon L^\infty(Q_T)\times X_1\times X_2\to\mathbb{R}^+$, $k_1\colon L^\infty(Q_T)\times X_1\times X_2\to L^{q_1}(Q_T)$ such that

$$\begin{aligned}|a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2)|\le \\ \le\mathfrak{c}_1(w,v_1,v_2)c_1(\xi)\left(|\zeta_0|^{p_1-1}+|\zeta|^{p_1-1}+|\eta_0|^{\frac{p_2}{q_1}}+|\eta|^{\frac{p_2}{q_1}}+[k_1(w,v_1,v_2)](t,x)\right),\end{aligned}$$

for a.a. $(t,x)\in Q_T$, every $(\xi,\zeta_0,\zeta,\eta_0,\eta)\in\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ and $(w,v_1,v_2)\in L^\infty(Q_T)\times X_1\times X_2$ $(i=0,\dots,n)$.

A3. There exists a constant $C>0$ such that for a.a. $(t,x)\in Q_T$, every $(\xi,\zeta_0,\zeta,\eta_0,\eta)$, $(\xi,\zeta_0,\tilde\zeta,\eta_0,\eta)\in\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ and $(w,v_1,v_2)\in L^\infty(Q_T)\times X_1\times X_2$

$$\sum_{i=1}^n\Big(a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2)-a_i(t,x,\xi,\zeta_0,\tilde\zeta,\eta_0,\eta;w,v_1,v_2)\Big)(\zeta_i-\tilde\zeta_i)\ge C\cdot|\zeta-\tilde\zeta|^{p_1}.$$

A4. There exist a constant $c_2>0$, a continuous function $\gamma\colon\mathbb{R}\to\mathbb{R}$ and bounded operators $\Gamma\colon L^\infty(Q_T)\to L^\infty(Q_T)$, $k_2\colon X_1\to L^1(Q_T)$ such that

$$\sum_{i=0}^n a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2)\zeta_i\ge c_2\left(|\zeta_0|^{p_1}+|\zeta|^{p_1}\right)-\gamma(\xi)[\Gamma(w)](t,x)[k_2(v_1)](t,x)$$

for a.a. $(t,x)\in Q_T$ and every $(\xi,\zeta_0,\zeta,\eta_0,\eta)\in\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$, $(w,v_1,v_2)\in L^\infty(Q_T)\times X_1\times X_2$. Further,

$$\lim_{\|v_1\|_{X_1}\to+\infty}\frac{\|k_2(v_1)\|_{L^1(Q_T)}}{\|v_1\|_{X_1}^{p_1}}=0. \tag{8}$$

A5. If (ω_k) is bounded in $L^\infty(Q_T)$, $\omega_k\to\omega$ a.e. in Q_T and $u_k\to u$ weakly in X_1, strongly in $L^{p_1}(Q_T)$, further, $\mathbf{p}_k\to\mathbf{p}$ strongly in X_2 then

$$\lim_{k\to\infty}\|a_i(\cdot,\omega_k,u_k,Du_k,\mathbf{p}_k,D\mathbf{p}_k;\omega_k,u_k,\mathbf{p}_k)-a_i(\cdot,\omega_k,u_k,Du_k,\mathbf{p}_k,D\mathbf{p}_k;\omega,u,\mathbf{p})\|_{L^{q_1}(Q_T)}=0.$$

B1. For fixed $(w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2$ functions $b_i \colon Q_T \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1} \times L^\infty(Q_T) \times X_1 \times X_2 \to \mathbb{R}$ $(i = 0, \dots, n)$ have the Carathéodory property, i.e., they are measurable in $(t, x) \in Q_T$ for every $(\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$ and continuous in $(\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$ for a.a. $(t, x) \in Q_T$.

B2. There exist a continuous function $\hat{c}_1 \colon \mathbb{R} \to \mathbb{R}^+$ and bounded operators $\hat{\mathfrak{c}}_1 \colon L^\infty(Q_T) \times X_1 \times X_2 \to \mathbb{R}^+$, $\hat{k}_1 \colon L^\infty(Q_T) \times X_1 \times X_2 \to L^{q_2}(Q_T)$ such that

$$|b_i(t, x, \xi, \zeta_0, \eta_0, \eta; w, v_1, v_2)| \leq \hat{\mathfrak{c}}_1(w, v_1, v_2) \hat{c}_1(\xi) \left(|\eta_0|^{p_2 - 1} + |\eta|^{p_2 - 1} + |\zeta_0|^{\frac{p_1}{q_2}} + [\hat{k}_1(w, v_1, v_2)](t, x) \right)$$

for a.a. $(t, x) \in Q_T$ and every $(\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$, $(w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2$ $(i = 0, \dots, n)$.

B3. There exists a constant $\hat{C} > 0$ such that for a.a. $(t, x) \in Q_T$, every $(\xi, \zeta_0, \eta_0, \eta)$, $(\xi, \zeta_0, \tilde{\eta}_0, \tilde{\eta}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$ and $(w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2$

$$\sum_{i=0}^{n} \left(b_i(t, x, \xi, \zeta_0, \eta_0, \eta; w, v_1, v_2) - b_i(t, x, \xi, \zeta_0, \tilde{\eta}_0, \tilde{\eta}; w, v_1, v_2) \right) (\eta_i - \tilde{\eta}_i) \geq \hat{C} \cdot \left(|\eta_0 - \tilde{\eta}_0|^{p_2} + |\eta - \tilde{\eta}|^{p_2} \right).$$

B4. There exist a constant $\hat{c}_2 > 0$, a continuous function $\hat{\gamma} \colon \mathbb{R} \to \mathbb{R}$ and bounded operators $\hat{\Gamma} \colon L^\infty(Q_T) \to L^\infty(Q_T)$, $\hat{k}_2 \colon X_2 \to L^1(Q_T)$ such that

$$\sum_{i=0}^{n} b_i(t, x, \xi, \zeta_0, \eta_0, \eta; w, v_1, v_2) \eta_i \geq \hat{c}_2 \left(|\eta_0|^{p_2} + |\eta|^{p_2} \right) - \hat{\gamma}(\xi) [\hat{\Gamma}(w)](t, x) \left(|\zeta_0|^{p_1} + [\hat{k}_2(v_2)](t, x) \right)$$

for a.a. $(t, x) \in Q_T$, and every $(\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$, $(w, v_1, v_2) \in L^\infty(Q_T) \times X_1 \times X_2$. Further,

$$\lim_{\|v_2\|_{X_2} \to \infty} \frac{\|\hat{k}_2(v_2)\|_{L^1(Q_T)}}{\|v_2\|_{X_2}^{p_2}} = 0. \tag{9}$$

B5. If (ω_k) is bounded in $L^\infty(Q_T)$, $\omega_k \to \omega$ a.e. in Q_T and $u_k \to u$ weakly in X_1, strongly in $L^{p_1}(Q_T)$, further, $\mathbf{p}_k \to \mathbf{p}$ weakly in X_2, strongly in $L^{p_2}(Q_T)$ then

$$\lim_{k \to \infty} \|b_i(\cdot, \omega_k, u_k, \mathbf{p}_k, D\mathbf{p}_k; \omega_k, u_k, \mathbf{p}_k) - b_i(\cdot, \omega_k, u_k, \mathbf{p}_k, D\mathbf{p}_k; \omega, u, \mathbf{p})\|_{L^{q_2}(Q_T)} = 0.$$

F1. For fixed $v \in X_1$ function $f \colon Q_T \times \mathbb{R}^2 \times L^\infty(Q_T) \times X_1 \to \mathbb{R}$ is a Carathéodory function, i.e., it is measurable in $(t, x) \in Q_T$ for every $(\xi, \zeta_0) \in \mathbb{R}^2$ and continuous in $(\xi, \zeta_0) \in \mathbb{R}^2$ for a.a. $(t, x) \in Q_T$. Further, there exists a bounded operator $\mathcal{K}_1 \colon X_1 \to \mathbb{R}^+$ such that

(i) for every bounded set $I \subset \mathbb{R}$ there is a continuous function $K_1 \colon \mathbb{R} \to \mathbb{R}^+$ satisfying $|K_1(\zeta_0)| \leq d_1 |\zeta_0|^{\frac{p_1}{q_2}} + d_2$ for every $\zeta_0 \in \mathbb{R}$, with some nonnegative constants d_1, d_2 (depending on I),

(ii) for a.a. $(t, x) \in Q_T$, every $(\xi, \zeta_0), (\tilde{\xi}, \zeta_0) \in I \times \mathbb{R}$ and every $v \in X_1$

$$|f(t, x, \xi, \zeta_0; v) - f(t, x, \tilde{\xi}, \zeta_0; v)| \leq \mathcal{K}_1(v) K_1(\zeta_0) \cdot |\xi - \tilde{\xi}|.$$

F2. There exist a bounded operator $\mathcal{K}_2 \colon X_1 \to \mathbb{R}^+$ and a continuous function $K_2 \colon \mathbb{R} \to \mathbb{R}^+$ such that for a.a. $(t, x) \in Q_T$, every (ξ, ζ_0), $(\xi, \tilde{\zeta}_0) \in \mathbb{R}^2$ and $v \in X_1$

$$|f(t, x, \xi, \zeta_0; v) - f(t, x, \xi, \tilde{\zeta}_0; v)| \leq \mathcal{K}_2(v) K_2(\xi) \cdot |\zeta_0 - \tilde{\zeta}_0|.$$

F3. There exists $\omega^* \in L^\infty(\Omega)$ such that for a.a. $(t, x) \in Q_T$, every $(\xi, \zeta_0) \in \mathbb{R}^2$ and $v \in X_1$,

$$(\xi - \omega^*(x)) \cdot f(t, x, \xi, \zeta_0; v) \leq 0.$$

F4. If (ω_k) is bounded in $L^\infty(Q_T)$ and $u_k \to u$ strongly in $L^{p_1}(Q_T)$ then

$$\lim_{k \to \infty} \|f(\cdot, \omega_k, u_k; u_k) - f(\cdot, \omega_k, u_k; u)\|_{L^1(Q_T)} = 0.$$

1.4 Weak form

If the above assumptions are satisfied we may define operators $A\colon L^\infty(Q_T)\times X_1\times X_2\to X_1^*$, $B\colon L^\infty(Q_T)\times X_1\times X_2\to X_2^*$ by:

$$[A(\omega,u,\mathbf{p}),v_1]:=\int_{Q_T}\sum_{i=1}^n a_i(t,x,\omega(t,x),u(t,x),Du(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p})D_iv_1(t,x)\,dt\,dx+$$
$$+\int_{Q_T}a_0(t,x,\omega(t,x),u(t,x),Du(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p})v_1(t,x)\,dt\,dx, \quad (10)$$

$$[B(\omega,u,\mathbf{p}),v_2]:=\int_{Q_T}\sum_{i=1}^n b_i(t,x,\omega(t,x),u(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p})D_iv_2(t,x)\,dt\,dx+$$
$$+\int_{Q_T}b_0(t,x,\omega(t,x),u(t,x),\mathbf{p}(t,x),D\mathbf{p}(t,x);\omega,u,\mathbf{p})v_2(t,x)\,dt\,dx, \quad (11)$$

for $v_1\in X_1$ and $v_2\in X_2$. In addition, let us introduce the linear operator $L:D(L)\to X_1^*$ by the formula

$$D(L)=\{u\in X_1\colon D_tu\in X_1^*,\ u(0)=0\},\qquad Lu=D_tu. \quad (12)$$

By the operators above we may define the weak form of system (5)–(7) as

$$\omega(t,x)=\omega_0(x)+\int_0^t f(s,x,\omega(s,x),u(s,x);u)\,ds\quad\text{for a.a. }(t,x)\in Q_T \quad (13)$$
$$Lu+A(\omega,u,\mathbf{p})=G \quad (14)$$
$$B(\omega,u,\mathbf{p})=H \quad (15)$$

where $G\in X_1^*$ and $H\in X_2^*$ are definde by

$$[G,v_1]=\int_{Q_T}g(t,x)v_1(t,x)\,dt\,dx,\quad [H,v_2]=\int_{Q_T}h(t,x)v_2(t,x)\,dt\,dx$$

where $v_i\in X_i$ $(i=1,2)$.

2 Existence of solutions

In this section we prove

Theorem 1. *Suppose that conditions A1–A5, B1–B5, F1–F4 are fulfilled. Then for every $\omega_0\in L^\infty(\Omega)$, $G\in X_1^*$ and $H\in X_2^*$ there exists a solution $\omega\in L^\infty(Q_T)$, $u\in D(L)$, $\mathbf{p}\in L^{p_2}(0,T;V_2)$ of problem (13)–(15).*

First we formulate some statements related to the solvability of the above equations (13)–(15).

Proposition 2. *Assume that conditions F1, F3 are satisfied. Then for every fixed $u\in L^{p_1}(Q_T)$ and $\omega_0\in L^\infty(Q_T)$ there exists a unique solution $\omega\in L^\infty(Q_T)$ of the integral equation (13), further, for the solution ω, estimate $\|\omega\|_{L^\infty(Q_T)}\le\|\omega_0\|_{L^\infty(\Omega)}+\|\omega^*\|_{L^\infty(\Omega)}$ holds.*

Proof. Immediately follows from Proposition 2.3 in [4] since for fixed nonlocal variable u, condition F1 is the same as in the cited paper. □

Proposition 3. *Assume F1–F4 and let $(u_k)\subset L^{p_1}(Q_T)$, further, let ω_k be the solution of (13) corresponding to u_k. If $u_k\to u$ in $L^{p_1}(Q_T)$ then $\omega_k\to\omega$ a.e. in Q_T where ω is the solution of (13) corresponding to u.*

Proof. Suppose that $u_k\to u$ in $L^{p_1}(Q_T)$ and let $\omega_k,\omega\in L^\infty(Q_T)$ be the corresponding solutions of (13). Then for a.a. $x\in\Omega$, $u_k(\cdot,x)\to u(\cdot,x)$. Fix such a point $x\in\Omega$. Since (ω_k) is bounded in $L^\infty(Q_T)$ by Proposition 2, we can apply condition F1, F2 and we obtain

$$|\omega_k(t,x)-\omega(t,x)|\le\int_0^t|f(s,x,\omega_k(s,x),u_k(s,x);u_k)-f(s,x,\omega_k(s,x),u_k(s,x);u)|\,ds+$$
$$\int_0^t|f(s,x,\omega_k(s,x),u_k(s,x);u)-f(s,x,\omega(s,x),u(s,x);u)|\,ds.$$

The first integral converges to 0 for a.a. $x \in \Omega$ by condition F4, further, by F1, F2 it is easy to show that the second integral is less then

$$\text{const} \cdot \left(\int_0^t |\omega_k(s,x) - \omega(s,x)|^{p_2} \, ds \right)^{1/p_2} + \text{const} \cdot \int_0^T |u_k(s,x) - u(s,x)| \, ds.$$

Hence

$$|\omega_k(t,x) - \omega(t,x)|^{p_2} \le \text{const} \cdot \int_0^t |\omega_k(s,x) - \omega(s,x)|^{p_2} \, ds + r(u_k)$$

where the remainder term $r(u_k)$ tends to 0 as $k \to \infty$. Thus Gronwall's lemma yields $|\omega_k(t,x) - \omega(t,x)| \le \text{const} \cdot r(u_k) \to 0$ which implies the desired a.e. convergence of (ω_k). $\square$

Proposition 4. *Assume A1–A5. Then for every fixed* $\omega \in L^\infty(Q_T)$, $\mathbf{p} \in X_2$ *and* $G \in X_1^*$ *there exists a solution* $u \in D(L)$ *of problem* $Lu + A(\omega, u, \mathbf{p}) = G$.

Proof. The proof follows from Theorem 1.1 in [13] (based on the theory of monotone type operators, see [2]) since for fixed $\omega \in L^\infty(Q_T)$ and $\mathbf{p} \in X_2$ conditions A1–A5 imply that operator $A(\omega, \cdot, \mathbf{p}) \colon X_1 \to X_1^*$ fulfils conditions I–V of the mentioned theorem. $\square$

Proposition 5. *Suppose that B1–B5 hold. Then for every fixed* $\omega \in L^\infty(Q_T)$, $u \in X_1$ *and* $H \in X_2^*$ *there exists a solution* $\mathbf{p} \in X_2$ *of problem* $B(\omega, u, \mathbf{p}) = H$.

Proof. The statement follows from the theory of monotone operators (see [15]) since conditions B1–B4 imply the boundedness, demicontinuity, uniform monotonicity and coerciveness of operator $B(\omega, u, \cdot) \colon X_2 \to X_2^*$ for fixed $\omega \in L^\infty(Q_T), u \in X_1$. $\square$

Proof of Theorem 1. The idea is similar as in [4]. We define sequences of approximate solutions of problem (13)–(15) and we show the boundedness of these sequences. After choosing weakly convergent subsequences we verify that the weak limits of the subsequences are solutions of the problem. For simplicity, in the proof we omit the variable (t,x) of functions a_i, b_i if it is not confusing.

Step 1: approximation. Define the sequences $(\omega_k), (u_k), (\mathbf{p}_k)$ as follows. Let $\omega_0(t,x) \equiv u_0(t,x) \equiv \mathbf{p}_0(t,x) \equiv 0$ $((t,x) \in Q_T)$ and for $k = 0, 1, \dots$ let $\omega_{k+1}, u_{k+1}, \mathbf{p}_{k+1}$ be a solution of the system:

$$\omega_{k+1}(t,x) = \omega_0(x) + \int_0^t f(s, x, \omega_{k+1}(s,x), u_k(s,x); u_k) \, ds \tag{16}$$

$$Lu_{k+1} + A(\omega_k, u_{k+1}, \mathbf{p}_k) = G \tag{17}$$

$$B(\omega_k, u_k, \mathbf{p}_{k+1}) = H. \tag{18}$$

By Propositions 2, 4, 5 we have solutions $\omega_{k+1} \in L^\infty(Q_T)$, $u_{k+1} \in X_1$, $\mathbf{p}_{k+1} \in X_2$ so the above recurrence yields the sequences $(\omega_k) \subset L^\infty(Q_T), (u_k) \subset X_1, (\mathbf{p}_k) \subset X_2$.

Step 2: boundedness. We show that the above defined sequences are bounded. By Proposition 2 for fixed $\omega_0 \in L^\infty(\Omega)$ for the solution of equation (16) estimate $\|\omega_{k+1}\|_{L^\infty(Q_T)} \le \|\omega_0\|_{L^\infty(\Omega)} + \|\omega^*\|_{L^\infty(\Omega)}$ holds thus (ω_k) is bounded in $L^\infty(Q_T)$

Now by choosing the test function $v = u_{k+1}$ in (17) and by using condition A4 and the monotonicity of operator L we obtain

$$[G, u_{k+1}] = [Lu_{k+1}, u_{k+1}] + [A(\omega_k, u_{k+1}, \mathbf{p}_k), u_{k+1}] \ge c_2 \int_{Q_T} (|u_{k+1}|^{p_1} + |Du_{k+1}|^{p_1} - \gamma(\omega_k)\Gamma(\omega_k) k_2(u_{k+1})) \ge$$

$$\ge c_2 \|u_{k+1}\|_{X_1} \left(\|u_{k+1}\|_{X_1}^{p_1 - 1} - \|\gamma(\omega_k)\Gamma(\omega_k)\|_{L^\infty(Q_T)} \cdot \frac{\|k_2(u_{k+1})\|_{L^1(Q_T)}}{\|u_{k+1}\|_{X_1}} \right),$$

thus by the boundedness of (ω_k) we conclude for some $K > 0$

$$\|u_{k+1}\|_{X_1}^{p_1 - 1} \left(1 - K \cdot \frac{\|k_2(u_{k+1})\|_{L^1(Q_T)}}{\|u_{k+1}\|_{X_1}^{p_1}} \right) \le \text{const}.$$

Now (8) implies that (u_k) is bounded in X_1.

The boundedness of $(\mathbf{p}_k)$ in X_2 follows by similar arguments as above by using condition B4 and the boundedness of the sequences (ω_k), (u_k).

We need also the boundedness of the sequence (Lu_k) in X_1^*. By Hölder's inequality

$$|[A(\omega_k, u_{k+1}, \mathbf{p}_k), v]| \le \left(\sum_{i=0}^n \|a_i(\omega_k, u_{k+1}, Du_{k+1}, \mathbf{p}_k, D\mathbf{p}_k; \omega_k, u_{k+1}, \mathbf{p}_k)\|_{L^{q_1}(Q_T)} \right) \cdot \|v\|_{X_1}.$$

and from condition A2 it follows that for all i

$$\|a_i(\omega_k, u_{k+1}, Du_{k+1}, \mathbf{p}_k, D\mathbf{p}_k; \omega_k, u_{k+1}, \mathbf{p}_k)\|_{L^{q_1}(Q_T)} \leq$$
$$\leq \text{const} \cdot c_1(\omega_k)\mathfrak{c}_1(\omega_k, u_{k+1}, \mathbf{p}_k)\left(\|u_{k+1}\|_{X_1}^{p_1} + \|\mathbf{p}_k\|_{X_2}^{p_2} + \|k_1(\omega_k, u_{k+1}, \mathbf{p}_k)\|_{L^{q_1}(Q_T)}\right).$$

Therefore by the boundedness of the sequences $(\omega_k), (u_k), (\mathbf{p}_k)$ and the boundedness of operators $c_1, \mathfrak{c}_1, k_2$ we conclude $|[Lu_{k+1}, v]| = |[A(\omega_k, u_{k+1}, \mathbf{p}_k) + G, v]| \leq \text{const} \cdot \|v\|_{X_1}$ so (Lu_k) is a bounded sequence in X_1^*.

Step 3: convergence. Due to the boundedness of the sequences $(u_k), (Lu_k), (\mathbf{p}_k)$ (in reflexive Banach spaces) each has a weakly convergent subsequence, further, by applying a well known embedding theorem (see [11]) it follows that there exist subsequences (which will be denoted same as the original sequences) and functions $\omega \in L^\infty(Q_T)$, $u \in X_1$, $\mathbf{p} \in X_2$ such that

$$u_k \to u \text{ weakly in } X_1, \text{ strongly in } L^{p_1}(Q_T), \text{ a.e. in } Q_T;$$
$$Lu_k \to Lu \text{ weakly in } X_1^*;$$
$$\mathbf{p}_k \to \mathbf{p} \text{ weakly in } X_2.$$

In what follows, we show that $\omega, u, \mathbf{p}$ are solutions of problem (13)–(15).

Since $u_k \to u$ in $L^{p_1}(Q_T)$, further, ω_{k+1} is the solution of equation (16), by Proposition 3 it follows that $\omega_k \to \omega$ a.e. in Q_T and functions ω, u satisfy the integral equation (13).

Now let us consider equation (18). First we show that $\mathbf{p}_k \to \mathbf{p}$ in X_2. To this end, let us introduce operator $\tilde{B}\colon L^\infty(Q_T) \times X_1 \times X_2 \times L^\infty(Q_T) \times X_1 \times X_2 \to X_2^*$ by

$$[\tilde{B}(\omega, u, \mathbf{p}; w, v_1, v_2), z_2] := \int_{Q_T} \sum_{i=1}^n b_i(t, x, \omega(t,x), u(t,x), \mathbf{p}(t,x), D\mathbf{p}(t,x); w, v_1, v_2) D_i z_2(t,x)\, dt\, dx +$$
$$+ \int_{Q_T} b_0(t, x, \omega(t,x), u(t,x), \mathbf{p}(t,x), D\mathbf{p}(t,x); w, v_1, v_2) z_2(t,x)\, dt\, dx \quad (19)$$

for $z_2 \in X_2$. Observe $B(\omega, u, \mathbf{p}) = \tilde{B}(\omega, u, \mathbf{p}; \omega, u, \mathbf{p})$. By condition B3 we have

$$[\tilde{B}(\omega_k, u_k, \mathbf{p}_{k+1}; \omega, u, \mathbf{p}) - \tilde{B}(\omega_k, u_k, \mathbf{p}; \omega, u, \mathbf{p}), \mathbf{p}_{k+1} - \mathbf{p}] \geq \hat{C} \cdot \|\mathbf{p}_{k+1} - \mathbf{p}\|_{X_2}^{p_2}. \quad (20)$$

On the left hand side of the above inequality we have the following decomposition:

$$[\tilde{B}(\omega_k, u_k, \mathbf{p}_{k+1}; \omega, u, \mathbf{p}) - \tilde{B}(\omega_k, u_k, \mathbf{p}; \omega, u, \mathbf{p}), \mathbf{p}_{k+1} - \mathbf{p}] = [\tilde{B}(\omega_k, u_k, \mathbf{p}_{k+1}; \omega_k, u_k, \mathbf{p}_{k+1}), \mathbf{p}_{k+1} - \mathbf{p}] +$$
$$+ [\tilde{B}(\omega_k, u_k, \mathbf{p}_{k+1}; \omega, u, \mathbf{p}) - \tilde{B}(\omega_k, u_k, \mathbf{p}_{k+1}; \omega_k, u_k, \mathbf{p}_{k+1}), \mathbf{p}_{k+1} - \mathbf{p}] +$$
$$+ [\tilde{B}(\omega, u, \mathbf{p}; \omega, u, \mathbf{p}) - \tilde{B}(\omega_k, u_k, \mathbf{p}; \omega, u, \mathbf{p}), \mathbf{p}_{k+1} - \mathbf{p}] - [\tilde{B}(\omega, u, \mathbf{p}; \omega, u, \mathbf{p}), \mathbf{p}_{k+1} - \mathbf{p}]. \quad (21)$$

We show that each term on the right hand side tends to 0. By recurrence (18), $\tilde{B}(\omega_k, u_k, \mathbf{p}_{k+1}; \omega_k, u_k, \mathbf{p}_{k+1}) = H$, further, $\mathbf{p}_{k+1} \to \mathbf{p}$ weakly in X_2 which implies the convergence of the first and the last term. The convergence of the second term follows from condition B5. In order to verify the convergence of the third term, observe that

$$|[\tilde{B}(\omega_k, u_k, \mathbf{p}; \omega, u, \mathbf{p}) - \tilde{B}(\omega, u, \mathbf{p}; \omega, u, \mathbf{p}), \mathbf{p}_{k+1} - \mathbf{p}]| \leq$$
$$\leq \sum_{i=0}^n \|b_i(\omega_k, u_k, \mathbf{p}, D\mathbf{p}; \omega, u, \mathbf{p}) - b_i(\omega, u, \mathbf{p}, D\mathbf{p}; \omega, u, \mathbf{p})\|_{L^{q_2}(Q_T)} \cdot \|\mathbf{p}_{k+1} - \mathbf{p}\|_{X_2} \quad (22)$$

and by condition B2

$$|b_i(\omega_k, u_k, \mathbf{p}, D\mathbf{p}; \omega, u, \mathbf{p}) - b_i(\omega, u, \mathbf{p}, D\mathbf{p}; \omega, u, \mathbf{p})|^{q_2} \leq$$
$$\leq \text{const} \cdot \hat{\mathfrak{c}}_1(\omega, u, \mathbf{p}) \cdot (|\hat{c}_1(\omega_k)|^{q_2} + |\hat{c}_1(\omega)|^{q_2}) \left(|\mathbf{p}|^{p_2} + |D\mathbf{p}|^{p_2} + |u_k|^{p_1} + |u|^{p_1} + |\hat{k}_1(\omega, u, \mathbf{p})|^{q_2}\right).$$

Due to the boundedness of (ω_k) in $L^\infty(Q_T)$ and the convergence of (u_k) in $L^{p_1}(Q_T)$ the left hand side of the above inequality is equi-integrable (see [8]), in addition, it a.e. converges to 0, therefore by Vitali's theorem the left hand side converges in $L^1(Q_T)$ to the zero function. Thus (because of the boundedness of $(\mathbf{p}_k)$) the right hand side of (22) tends to 0. Hence all terms on the right hand side of equation (21) converges to 0 thus (20) implies $\mathbf{p}_{k+1} \to \mathbf{p}$ in X_2.

Now by using the same arguments as in [4] one obtains that $\tilde{B}(\omega_k, u_k, \mathbf{p}_{k+1}; \omega, u, \mathbf{p}) \to \tilde{B}(\omega, u, \mathbf{p}; \omega, u, \mathbf{p}) = B(\omega, u, \mathbf{p})$ weakly in X_2^*. Further, by condition B5 it is not difficult to see that $\tilde{B}(\omega_k, u_k, \mathbf{p}_{k+1}; \omega, u, \mathbf{p}) - \tilde{B}(\omega_k, u_k, \mathbf{p}_{k+1}; \omega_k, u_k, \mathbf{p}_{k+1}) \to 0$ strongly in X_2^* thus $\tilde{B}(\omega_k, u_k, \mathbf{p}_{k+1}; \omega_k, u_k, \mathbf{p}_{k+1}) \to B(\omega, u, \mathbf{p})$. Then from recurrence (18) we conclude $B(\omega, u, \mathbf{p}) = H$, i.e., $\omega, u, \mathbf{p}$ are solutions of problem (15).

Finally, $A(\omega, u, \mathbf{p}) = G$ can be shown by similar arguments as above. The proof of the theorem is complete. □

3 Examples

We show some examples for functions satisfying conditions A1–A5, B1–B5. Let functions a_i, b_i have the form

$$a_i(t, x, \xi, \zeta_0, \zeta, \eta_0, \eta; w, v_1, v_2) = [\pi(w)](t, x)[\varphi(v_1)](t, x)[\psi(v_2)](t, x)P(\xi)Q(\eta_0, \eta)\zeta_i|\zeta|^{p_1-2}+ \\ + [\tilde{\pi}(w)](t, x)[\tilde{\varphi}(v_1)](t, x)\tilde{P}(\xi)\zeta_i|\zeta|^{r_1-1}, \text{ if } i \neq 0, \quad (23)$$

$$a_0(t, x, \xi, \zeta_0, \zeta, \eta_0, \eta; w, v_1, v_2) = [\pi(w)](t, x)[\varphi(v_1)](t, x)[\psi(v_2)](t, x)P(\xi)Q(\eta_0, \eta)\zeta_0|\zeta_0|^{p_1-2}+ \\ + [\tilde{\pi}_0(w)](t, x)[\tilde{\varphi}_0(v_1)](t, x)\tilde{P}_0(\xi)\zeta_0|\zeta_0|^{r_1-1}, \quad (24)$$

$$b_i(t, x, \xi, \zeta_0, \eta_0, \eta; w, v_1, v_2) = [\kappa(w)](t, x)[\lambda(v_1)](t, x)[\vartheta(v_2)](t, x)R(\xi)S(\zeta_0)\eta_i|(\eta_0, \eta)|^{p_2-2}+ \\ + [\tilde{\kappa}(w)](t, x)[\tilde{\vartheta}(v_2)](t, x)\tilde{R}(\xi)\eta_i|(\eta_0, \eta)|^{r_2-1}, \ i = 0, \dots, n, \quad (25)$$

where $1 \le r_i < p_i - 1$ $(i = 1, 2)$ and the following hold.

E1. a) Operators $\pi\colon L^\infty(Q_T) \to L^\infty(Q_T)$, $\varphi\colon L^{p_1}(Q_T) \to L^\infty(Q_T)$, $\psi\colon X_2 \to L^\infty(Q_T)$ are bounded, φ and ψ are continuous, further, if (ω_k) is bounded in $L^\infty(Q_T)$ and $\omega_k \to \omega$ a.e. in Q_T then $\pi(\omega_k) \to \pi(\omega)$ in $L^\infty(Q_T)$. In addition, $P \in C(\mathbb{R})$, $Q \in C(\mathbb{R}^{n+1}) \cap L^\infty(\mathbb{R}^{n+1})$, and there exists a positive lower bound for the values of π, φ, ψ, P, Q.

b) Operators $\tilde{\pi}, \tilde{\pi}_0\colon L^\infty(Q_T) \to L^\infty(Q_T)$, $\tilde{\varphi}, \tilde{\varphi}_0\colon L^{p_1}(Q_T) \to L^{\frac{p_1-1}{p_1-r_1-1}}(Q_T)$ are bounded, $\tilde{\varphi}$ and $\tilde{\varphi}_0$ are continuous, further, if (ω_k) is bounded in $L^\infty(Q_T)$ and $\omega_k \to \omega$ a.e. in Q_T then $\tilde{\pi}(\omega_k) \to \tilde{\pi}(\omega)$ and $\tilde{\pi}_0(\omega_k) \to \tilde{\pi}_0(\omega)$ in $L^\infty(Q_T)$. In addition, $\tilde{P}, \tilde{P}_0 \in C(\mathbb{R})$, operators $\tilde{\pi}, \tilde{\varphi}$ and function $\tilde{P}$ are nonnegative and

$$\lim_{\|v_1\|_{X_1} \to +\infty} \frac{\int_{Q_T} |\tilde{\varphi}_0(v_1)|^{\frac{p_1-1}{p_1-r_1-1}}}{\|v_1\|_{X_1}^{p_1}} = 0.$$

E2. a) Operators $\kappa\colon L^\infty(Q_T) \to L^\infty(Q_T)$, $\lambda\colon L^{p_1}(Q_T) \to L^\infty(Q_T)$, $\vartheta\colon L^{p_2}(Q_T) \to L^\infty(Q_T)$ are bounded, λ and ϑ are continuous, further, if (ω_k) is bounded in $L^\infty(Q_T)$ and $\omega_k \to \omega$ a.e. in Q_T then $\kappa(\omega_k) \to \kappa(\omega)$ in $L^\infty(Q_T)$. In addition, $R \in C(\mathbb{R})$, $S \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and there exists a positive lower bound for the values of $\kappa, \lambda, \vartheta, R, S$.

b) Operators $\tilde{\kappa}\colon L^\infty(Q_T) \to L^\infty(Q_T)$, $\tilde{\vartheta}\colon L^{p_2}(Q_T) \to L^{\frac{p_2-1}{p_2-r_2-1}}(Q_T)$ are bounded, $\tilde{\vartheta}$ is continuous, function $\tilde{R} \in C(\mathbb{R})$, further, if (ω_k) is bounded in $L^\infty(Q_T)$ and $\omega_k \to \omega$ a.e. in Q_T then $\tilde{\kappa}(\omega_k) \to \tilde{\kappa}(\omega)$ in $L^\infty(Q_T)$. In addition, operators $\tilde{\kappa}, \tilde{\vartheta}$ and function $\tilde{R} \in C(\mathbb{R})$ are nonnegative and

$$\lim_{\|v_2\|_{X_2} \to +\infty} \frac{\int_{Q_T} |\tilde{\vartheta}(v_2)|^{\frac{p_2-1}{p_2-r_2-1}}}{\|v_2\|_{X_2}^{p_2}} = 0.$$

Proposition 6. *Assume that E1-E2 hold, then functions* (23)–(25) *fulfil conditions A1–A5, B1–B5.*

By using Young's and Hölder's inequalities it is not difficult to prove the above statement, a detailed proof can be found in [3].

Operators $\pi, \tilde{\pi}, \tilde{\pi}_0, \kappa, \tilde{\kappa}$ may have the form $[\pi(w)](t, x) = \displaystyle\int_{Q_t} |w|^\beta$, where $1 \le \beta$. Further, operators φ, λ may have one of the forms

$$[\varphi(v)](t, x) = \Phi\left(\int_{Q_t} |v|^\beta\right) \quad \text{or} \quad \Phi\left(\int_{Q_t} dv\right),$$

where $1 \le \beta \le p_1$, $d \in L^{q_1}(Q_T)$, $\Phi \in C(\mathbb{R})$ and $\Phi \ge \text{const} > 0$. Similarly, ψ may have in the form

$$[\psi(v)](t, x) = \Psi\left(\int_{Q_t} |v|^\beta + |Dv|^\beta\right) \quad \text{or} \quad \Psi\left(\int_{Q_t} d_1 v + d_2|Dv|\right),$$

where $1 \le \beta \le p_2$, $d_1, d_2 \in L^{q_2}(Q_T)$, $\Psi \in C(\mathbb{R})$ and $\Psi \ge \text{const} > 0$ For $\tilde{\varphi}$ consider, e.g.,

$$[\tilde{\varphi}(v)](t, x) = \tilde{\Phi}\left(\int_0^t d(s, x)v(s, x)\, ds\right), \quad \tilde{\Phi}\left(\int_\Omega d(t, x)v(t, x)\, dx\right) \quad \text{or} \quad \tilde{\Phi}\left(\left[\int_0^t |v(s, x)|^\beta\, ds\right]^{\frac{1}{\beta}}\right),$$

where $d \in L^\infty(Q_T)$, $1 \le \beta \le p_1$, $\tilde{\Phi} \in C(\mathbb{R})$, $\tilde{\Phi} \ge 0$ and $|\tilde{\Phi}(\tau)| \le \text{const} \cdot |\tau|^{p_1-r_1-1}$. In the case of $\tilde{\varphi}_0$ one has similar examples as for $\tilde{\varphi}$ above, except $\tilde{\Phi}$ does not have to be nonnegative.

For operators $\vartheta, \tilde{\vartheta}$ we may consider similar examples as for $\varphi, \tilde{\varphi}$ above, by replacing exponents p_1 with p_2 and r_1 with r_2.

It is not difficult to show that the above operators fulfil conditions E1–E2, for similar arguments see, e.g., [3].

As an example for function f consider, e.g., $f(t,x,\xi,\zeta_0;v) = -[\varphi(v)](t,x)f_1(t,x)f_2(\zeta_0)(\xi - \omega^*(x))$, where $\varphi\colon L^{p_1}(Q_T) \to L^\infty(Q_T)$ is bounded and nonnegative, further, $f_1 \in L^\infty(Q_T)$, $f_2\colon \mathbb{R} \to \mathbb{R}$ is nonnegative, Lipschitz continuous and $|f_2(\zeta_0)| \le \text{const}|\zeta_0|^{\frac{p_1}{q_2}}$.

4 Solutions in $(0,\infty)$

In the previous section we have proved existence of solutions for all finite time interval $(0,T)$. In what follows we shall show existence of weak solutions in $(0,\infty)$. Denote by $X_i^\infty = L^{p_i}_{\text{loc}}(0,\infty;V_i)$ the space of measurable functions $u\colon (0,\infty) \to V_i$ such that $u|_{(0,T)} \in L^{p_i}(0,T;V_i)$ for every $0 < T < \infty$, further, let $L^\infty_{\text{loc}}(Q_\infty)$ be the space of functions $\omega\colon Q_\infty \to \mathbb{R}$ such that $\omega|_{Q_T} \in L^\infty(Q_T)$ for every $0 < T < \infty$. In the following we suppose

(Vol) Functions $a_i\colon Q_\infty \times \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times L^\infty(Q_\infty) \times X_1^\infty \times X_2^\infty \to \mathbb{R}$, $b_i\colon Q_\infty \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1} \times L^\infty(Q_\infty) \times X_1^\infty \times X_2^\infty \to \mathbb{R}$ $(i = 0,\dots,n)$, $f\colon Q_\infty \times \mathbb{R}^2 \times L^\infty_{\text{loc}}(Q_\infty) \times X_1^\infty \to \mathbb{R}$ have the Volterra property, i.e., $a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2)|_{Q_T}$, $b_i(t,x,\xi,\zeta_0,\eta_0,\eta;w,v_1,v_2)|_{Q_T}$, $f(t,x,\xi,\zeta_0;w)|_{Q_T}$ depend only on $(w|_{Q_T}, v_1|_{Q_T}, v_2|_{Q_T})$ for every $0 < T < \infty$.

Now we may define the weak form of (5)–(7) in Q_∞. For fixed $0 < T < \infty$ introduce operators $A_T\colon L^\infty(Q_T) \times L^{p_1}(0,T;V_1) \times L^{p_2}(0,T;V_2) \to L^{q_1}(0,T;V_1^*)$, $B_T\colon L^\infty(Q_T) \times L^{p_1}(0,T;V_1) \times L^{p_2}(0,T;V_2) \to L^{q_2}(0,T;V_2^*)$, $L_T\colon D(L_T) \to L^{q_i}_{\text{loc}}(0,T;V_i^*)$ by formulas (10)–(12). We say that $\omega \in L^\infty_{\text{loc}}(Q_\infty), u \in X_1^\infty, \mathbf{p} \in X_2^\infty$ is a solution of (5)–(7) in $(0,\infty)$ if for all $0 < T < \infty$ (for the restrictions of the functions to Q_T)

$$\omega(t,x) = \omega_0(x) + \int_0^t f(s,x,\omega(s,x),u(s,x);u)\,ds \quad (t,x) \in Q_T \tag{26}$$

$$L_T u + A_T(\omega,u,\mathbf{p}) = G_T \tag{27}$$

$$B_T(\omega,u,\mathbf{p}) = H_T \tag{28}$$

where $G_T = G|_{(0,T)}, H_T = H|_{(0,T)}$ with $G \in L^{q_1}_{\text{loc}}(0,\infty;V_1^*), H \in L^{q_2}_{\text{loc}}(0,\infty;V_2^*)$. Observe that the Volterra property ensures that if $\omega, u, \mathbf{p}$ is a solution in $(0,T)$ for some T then these functions are solutions in $(0,\tilde{T})$ for all $\tilde{T} < T$.

Theorem 7. *Suppose that (Vol), A1–A5, B1–B5, F1–F4 hold (in the sense that they are satisfied by the restrictions of functions a_i, b_i, f to Q_T for all $0 < T < \infty$). Then there exists $\omega \in L^\infty(Q_\infty), u \in X_1^\infty, \mathbf{p} \in X_2^\infty$ such that $\omega|_{Q_T}, u|_{Q_T}, \mathbf{p}|_{Q_T}$ is a solution of problem (13)–(15) for all $0 < T < \infty$.*

Idea of the proof. One may apply the arguments of the proof of Theorem 1 in [6] word for word. The idea is the following. Due to Theorem 1 we have solutions in $(0,T_k)$ where $T_k \to \infty$. By showing the boundedness of these solutions and using a diagonal process one may choose weakly convergent sequences of solutions. After taking $k \to \infty$ we obtain a solution in $(0,\infty)$. □

Remark 8. *Examples (23)–(25) fulfil the conditions of the above theorem if operators $\pi, \tilde{\pi}, \tilde{\pi}_0, \kappa, \tilde{\kappa}\colon L^\infty_{\text{loc}}(Q_\infty) \to L^\infty_{\text{loc}}(Q_\infty)$, $\varphi, \lambda\colon L^{p_1}_{\text{loc}}(Q_\infty) \to L^\infty_{\text{loc}}(Q_\infty)$, $\psi, \vartheta\colon L^{p_2}_{\text{loc}}(Q_\infty) \to L^\infty_{\text{loc}}(Q_\infty)$, $\tilde{\varphi}, \tilde{\varphi}_0\colon L^{p_1}_{\text{loc}}(Q_\infty) \to L^{\frac{p_1-1}{p_1-r_1-1}}_{\text{loc}}(Q_\infty)$, $\tilde{\vartheta}\colon L^{p_2}_{\text{loc}}(Q_\infty) \to L^{\frac{p_2-1}{p_2-r_2-1}}_{\text{loc}}(Q_\infty)$ are of Volterra type and conditions E1–E2 are satisfied for all finite $T > 0$. E.g., the operators given after Proposition 6 serve as an example for the above.*

4.1 Boundedness

Now we show that under some further assumptions, the solutions, formulated in the previous theorem are bounded in appropriate norms in the time interval $(0,\infty)$. First suppose

A4*. There exist a constant $c_2 > 0$, a continuous function $\gamma\colon \mathbb{R} \to \mathbb{R}$ and Volterra operators $\Gamma\colon L^\infty_{\text{loc}}(Q_\infty) \to L^\infty_{\text{loc}}(Q_\infty)$, $k_2\colon X_1^\infty \to L^1_{\text{loc}}(Q_\infty)$ such that

$$\sum_{i=0}^n a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2)\zeta_i \ge c_2\left(|\zeta_0|^{p_2} + |\zeta|^{p_2}\right) - \gamma(\xi)[\Gamma(w)](t,x)[k_2(v_1)](t,x)$$

for a.a. $(t,x) \in Q_\infty$, every $(\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$ and $(w, v_1, v_2) \in L^\infty(Q_\infty) \times X_1^\infty \times X_2^\infty$. Further, (8) holds and for all $v \in X_1^\infty$ uniformly in T

$$\int_\Omega |[k_2(v_1)](t,x)| \, dx \le \alpha_1 \left[\sup_{\tau \in [0,t]} \|v_1(\tau)\|_{L^2(\Omega)}^{\rho_1} + \chi_1(t) \sup_{\tau \in [0,t]} \|v_1(\tau)\|_{L^2(\Omega)}^{p_1} + 1 \right]$$

with some $\alpha_1 > 0, \rho_1 < p_1$ and $\chi_1 \colon \mathbb{R} \to \mathbb{R}$ such that $\lim_{t\to\infty} \chi_1(t) = 0$.

B4*. There exist a constant $\hat{c}_2 > 0$, a continuous function $\hat{\gamma} \colon \mathbb{R} \to \mathbb{R}$ and Volterra operators $\hat{\Gamma} \colon L^\infty(Q_\infty) \to L^\infty(Q_\infty)$, $k_2 \colon X_2^\infty \to L^1_{\mathrm{loc}}(Q_\infty)$ such that

$$\sum_{i=0}^n b_i(t,x,\xi,\zeta_0,\eta_0,\eta; w, v_1, v_2)\eta_i \ge \hat{c}_2 \left(|\eta_0|^{p_2} + |\eta|^{p_2}\right) - \hat{\gamma}(\xi)[\hat{\Gamma}(w)](t,x) \left(|\zeta_0|^2 + [\hat{k}_2(v_2)](t,x)\right)$$

for a.a. $(t,x) \in Q_\infty$, every $(\xi, \zeta_0, \eta_0, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n+1}$ and $(w, v_1, v_2) \in L^\infty(Q_\infty) \times X_1^\infty \times X_2^\infty$. Further, (9) holds and for all $v \in X_2^\infty$ uniformly in T

$$\int_\Omega |[\hat{k}_2(v_2)](t,x)| \, dx \le \alpha_2 \left[\operatorname*{ess\,sup}_{\tau \in [0,t]} \|v_2(\tau)\|_{V_2}^{\rho_2} + \chi_2(t) \operatorname*{ess\,sup}_{\tau \in [0,t]} \|v_2(\tau)\|_{V_2}^{p_2} + 1 \right]$$

with some $\alpha_2 > 0, \rho_2 < p_2$ and $\chi_2 \colon \mathbb{R} \to \mathbb{R}$ such that $\lim_{t\to\infty} \chi_2(t) = 0$.

Theorem 9. *Suppose $p_1, p_2 > 2$ and conditions (Vol), A1–A3, A4*, A5, B1–B3, B4*, B5, F1–F4 are satisfied, further, $\|G(\cdot)\|_{V_1^*}, \|H(\cdot)\|_{V_2^*} \in L^\infty(0,\infty)$. Then for the solutions $\omega, u, \mathbf{p}$ formulated in Theorem 9, $\omega \in L^\infty(Q_\infty)$, $u \in L^\infty(0,\infty; L^2(\Omega)), \mathbf{p} \in L^\infty(0,\infty; V_2)$.*

Idea of the proof. We may apply the arguments of the proof of Theorem 2 in [6]. Introduce the notation $y(t) = \|u(t,\cdot)\|^2_{L^2(\Omega)}$ (then y is continuous see, e.g, [15]). By choosing arbitrary $0 < T_1 < T_2 < \infty$, (13), conditions A4*, G2, Young's inequality and the continuous embedding $V_1 \hookrightarrow L^2(\Omega)$ imply

$$\frac{1}{2}\left(y(T_1) - y(T_2)\right) + \frac{1}{2} c_2 \int_{T_1}^{T_2} y(t)^{\frac{p_1}{2}} \, dt \le \mathrm{const} \cdot \int_{T_1}^{T_2} \left(\sup_{\tau \in [0,t]} y(\tau)^{\frac{\varrho_1}{2}} + \chi_1(t) \sup_{\tau \in [0,t]} y(\tau)^{\frac{p_1}{2}} + 1 \right) dt.$$

It is not difficult to see that the above inequality implies the boundedness of y in $(0,\infty)$, a detailed argument can be found in [14]. The boundedness of $\mathbf{p}$ follows from the boundedness of y similarly as above, by using conditions B4*, see [6]. □

Remark 10. *Example* (23)–(25) *fulfil the conditions of Theorem 9 if assumptions of Remark 8 are satisfied, in addition*

$$\int_\Omega |[\tilde{\varphi}_0(v_1)](t,x)|^{\frac{p_1-1}{p_1-r_1-1}} \, dx \le \alpha_1 \left[\sup_{\tau \in [0,t]} \|v_1(\tau)\|_{L^2(\Omega)}^{\rho_1} + \chi_1(t) \sup_{\tau \in [0,t]} \|v_1(\tau)\|_{L^2(\Omega)}^{p_1} + 1 \right]$$

for all $v_1 \in L^{p_1}_{\mathrm{loc}}(Q_\infty)$ with some constants $\alpha_1 > 0, \varrho_1 < p_1$ and function $\chi_1 \colon \mathbb{R} \to \mathbb{R}$ such that $\lim_{t\to\infty} \chi_1(t) = 0$, further, similar condition holds for $\tilde{\vartheta}$ (by changing the indeces from 1 to 2, and $L^2(\Omega)$ to V_2). For example, operator $\tilde{\varphi}_0$ may have the form

$$[\tilde{\varphi}(v)](t,x) = \tilde{\Phi}\left(\int_\Omega d(t,x)v(t,x)\,dx\right), \ \tilde{\Phi}\left(\int_\Omega |d(t,x)||v(t,x)|^\beta \, dx\right) \ \text{or} \ \chi_1(t)\tilde{\Phi}_0\left(\left[\int_\Omega |d(t,x)||v(t,x)|^2\,dx\right]^{\frac{1}{\beta}}\right),$$

where $d \in L^\infty(Q_\infty)$, $1 \le \beta \le 2$, $\tilde{\Phi}, \tilde{\Phi}_0, \chi_1 \in C(\mathbb{R})$ and $|\tilde{\Phi}(\tau)| \le \mathrm{const} \cdot |\tau|^{p_1-\varrho_1-1}$, $|\tilde{\Phi}_0(\tau)| \le \mathrm{const} \cdot |\tau|^{p_1-r_1-1}$, $\lim_{\tau\to\infty} \chi_1(\tau) = 0$.

4.2 Stabilization

In this section we consider a special case of problem (26)–(28), namely, let $p_1 = p_2 = p$ (thus $q_1 = q_2 = q$, $V_1 = V_2 = V$ and $X_1 = X_2 = X$). In what follows, we prove stabilization of the solutions of the system, that is, we show the convergence (in some sense) of solutions as $t \to \infty$ to the solutions of a stationary system. We need some further assumptions:

A6. There exist Carathéodory functions $a_{i,\infty}\colon \Omega\times\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}\to\mathbb{R}$ $(i=0,\dots,n)$ such that for a.a. $x\in\Omega$ and every $(\zeta_0,\zeta,\eta_0,\eta)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n$, $\xi^*\in\mathbb{R}$, $w\in L^\infty(Q_\infty)$, $v_1\in X^\infty\cap L^\infty(0,\infty;L^2(\Omega))$, $v_2\in X^\infty\cap L^\infty(0,\infty;V)$,

$$\lim_{\substack{t\to\infty\\ \xi\to\xi^*}} a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2)=a_{i,\infty}(x,\xi^*,\zeta_0,\zeta,\eta_0,\eta).$$

B6. There exist Carathéodory functions $b_{i,\infty}\colon \Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{n+1}\to\mathbb{R}$ $(i=0,\dots,n)$ such that for a.a. $x\in\Omega$ and every $(\zeta_0,\eta_0,\eta)\in\mathbb{R}\times\mathbb{R}^{n+1}$, $\xi^*\in\mathbb{R}$, $w\in L^\infty(Q_\infty)$, $v_1\in X^\infty\cap L^\infty(0,\infty;L^2(\Omega))$, $v_2\in X^\infty\cap L^\infty(0,\infty;V)$,

$$\lim_{\substack{t\to\infty\\ \xi\to\xi^*}} b_i(t,x,\xi,\zeta_0,\eta_0,\eta;w,v_1,v_2)=b_{i,\infty}(x,\xi^*,\zeta_0,\eta_0,\eta).$$

AB There exists a positive constant $\mathcal{C}$ such that for a.a. $(t,x)\in Q_\infty$ and every $\xi\in\mathbb{R}$, $(\zeta_0,\zeta,\eta_0,\eta)$, $(\tilde\zeta_0,\tilde\zeta,\tilde\eta_0,\tilde\eta)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n$, $(w,v_1,v_2)\in L^\infty(Q_\infty)\times X_1^\infty\times X_2^\infty$,

$$\sum_{i=0}^{n}\Big(a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta;w,v_1,v_2)-a_i(t,x,\xi,\tilde\zeta_0,\tilde\zeta,\tilde\eta_0,\tilde\eta;w,v_1,v_2)\Big)(\zeta_i-\tilde\zeta_i)+$$
$$+\sum_{i=0}^{n}\Big(b_i(t,x,\xi,\zeta_0,\eta_0,\eta)-b_i(t,x,\xi,\tilde\zeta_0,\tilde\eta_0,\tilde\eta)\Big)(\eta_i-\tilde\eta_i)\geq\mathcal{C}\cdot\Big(|\zeta_0-\tilde\zeta_0|^p+|\zeta-\tilde\zeta|^p+|\eta_0-\tilde\eta_0|^p+|\eta-\tilde\eta|^p\Big).$$

F5. For every fixed $v\in X^\infty\cap L^\infty(0,\infty;L^2(\Omega))$ there is a constant $m>0$ such that $(\xi-\omega^*(x))f(t,x,\xi,\zeta_0;v)\leq -m(\xi-\omega^*(x))^2$ for a.a. $(t,x)\in Q_\infty$ and every $(\xi,\zeta_0)\in\mathbb{R}^2$.

Now introduce operators $A_\infty\colon L^\infty(\Omega)\times V\times V\to V^*$, $B_\infty\colon L^\infty(\Omega)\times V\times V\to V^*$ by

$$\langle A_\infty(\omega,u,\mathbf{p}),v\rangle:=\int_\Omega\sum_{i=1}^{n}a_{i,\infty}(x,\omega(x),u(x),Du(x),\mathbf{p}(x),D\mathbf{p}(x))D_iv(x)\,dx+$$
$$+\int_\Omega a_{0,\infty}(x,\omega(x),u(x),Du(x),\mathbf{p}(x),D\mathbf{p}(x))v(x)\,dx,$$

$$\langle B_\infty(\omega,u,\mathbf{p}),v\rangle:=\int_\Omega\sum_{i=1}^{n}b_{i,\infty}(x,\omega(x),u(x),\mathbf{p}(x),D\mathbf{p}(x))D_iv(x)\,dx+$$
$$+\int_\Omega b_{0,\infty}(x,\omega(x),u(x),\mathbf{p}(x),D\mathbf{p}(x))v(x)\,dx,$$

Theorem 11. *Assume conditions A1–A3, A4*, A5–A6, B1–B3, B4*, B5–B6, AB, F1–F5 are satisfied (with $p=p_1=p_2$), further, there exist $F_\infty,G_\infty\in V^*$ such that*

$$\lim_{t\to\infty}\|F(t)-F_\infty\|_{V^*}=0,\quad \lim_{t\to\infty}\|G(t)-G_\infty\|_{V^*}=0.$$

Then there exist $u_\infty\in V,\mathbf{p}_\infty\in V$ such that for the solutions $\omega,u,\mathbf{p}$ of problem (26)–(28)*, $\omega(t,\cdot)\to\omega^*$ in $L^\infty(\Omega)$, $u(t)\to u_\infty$ in $L^2(\Omega)$, $\int_{t-1}^{t+1}\|u(s)-u_\infty\|_V^p\,ds\to 0$, $\int_{t-1}^{t+1}\|\mathbf{p}(s)-\mathbf{p}_\infty\|_V^p\,ds\to 0$, further,*

$$A_\infty(\omega^*,u_\infty,\mathbf{p}_\infty)=G_\infty \tag{29}$$
$$B_\infty(\omega^*,u_\infty,\mathbf{p}_\infty)=H_\infty. \tag{30}$$

Sketch of the proof. We follow the proof of Theorem 3 in [6]. Let $\omega,u,\mathbf{p}$ be solutions of (26)–(28) then by Theorem 9 $\omega\in L^\infty(Q_\infty)$, $u\in L^\infty(0,\infty;L^2(\Omega))$, $\mathbf{p}\in L^\infty(0,\infty;V_2)$. By using the same arguments as in the above mentioned paper, conditions F3, F5 imply estimate $\|\omega(t,\cdot)-\omega^*(\cdot)\|_{L^\infty(\Omega)}\leq\|\omega_0\|_{L^\infty(\Omega)}e^{-mt}$ which yields the convergence $\omega(t,\cdot)\to\omega^*(\cdot)$ in $L^\infty(\Omega)$.

Now by the using the idea of Proposition 5 and condition AB it is easy to see that for fixed ω^* there exist a unique solution $u_\infty\in V,\mathbf{p}_\infty\in V$ of problem (29)–(30) see, e.g., [15].

In order to show the desired convergences we prove a differential inequality for u and $\mathbf{p}$. From equations (26)–(28) and (29)–(30) we obtain

$$\langle D_t(u(t)-u_\infty),u(t)-u_\infty\rangle+\langle[A(\omega,u,\mathbf{p})](t)-A_\infty(\omega^*,u_\infty,\mathbf{p}_\infty),u(t)-u_\infty\rangle+$$
$$+\langle[B(\omega,u,\mathbf{p})](t)-B_\infty(\omega^*,u_\infty,\mathbf{p}_\infty),\mathbf{p}(t)-\mathbf{p}_\infty\rangle=\langle G(t)-G_\infty,u(t)-u_\infty\rangle+\langle F(t)-F_\infty,\mathbf{p}(t)-\mathbf{p}_\infty\rangle. \tag{31}$$

Observe that the first term equals to $\frac{1}{2}y'(t)$ where $y(t) = \int_\Omega (u(t) - u_\infty)^2$. Further, for the second and third terms of the above equation we have by condition AB and Young's inequality

$$\begin{aligned}
&\langle [A(\omega, u, \mathbf{p})](t) - A_\infty(\omega^*, u_\infty, \mathbf{p}_\infty), u(t) - u_\infty\rangle + \langle [B(\omega, u, \mathbf{p})](t) - B_\infty(\omega^*, u_\infty, \mathbf{p}_\infty), \mathbf{p}(t) - \mathbf{p}_\infty\rangle \geq \\
&\geq \mathcal{C} \cdot (\|u(t) - u_\infty\|_V^p + \|\mathbf{p}(t) - \mathbf{p}_\infty\|_V^p) - \frac{\varepsilon^p}{p}\|u(t) - u_\infty\|_V^p - \frac{\varepsilon^p}{p}\|\mathbf{p}(t) - \mathbf{p}_\infty\|_V^p - \\
&- \frac{1}{q\varepsilon^q}\|[\tilde{A}(\omega, u_\infty, \mathbf{p}_\infty; \omega, u, \mathbf{p})](t) - A_\infty(\omega^*, u_\infty, \mathbf{p}_\infty)\|_{V^*}^q - \\
&- \frac{1}{q\varepsilon^q}\|[\tilde{B}(\omega, u_\infty, \mathbf{p}_\infty; \omega, u, \mathbf{p})](t) - B(\omega^*, u_\infty, \mathbf{p}_\infty)\|_{V^*}^q
\end{aligned} \tag{32}$$

with some $\varepsilon > 0$ (and operator $\tilde{A}$ is defined similarly as $\tilde{B}$, see (19)). We show that last two terms on the right hand side of the above inequality converges to 0 as $t \to \infty$. Clearly,

$$\begin{aligned}
&\|[\tilde{A}(\omega, u_\infty, \mathbf{p}_\infty; \omega, u, \mathbf{p})](t) - A_\infty(\omega^*, u_\infty, \mathbf{p}_\infty)\|_{V^*}^q \leq \\
&\leq \sum_{i=0}^n \int_\Omega |a_i(t, \cdot, \omega(t, \cdot), u_\infty, Du_\infty, \mathbf{p}_\infty, D\mathbf{p}_\infty; \omega, u, \mathbf{p}) - a_{i,\infty}(\omega^*, u_\infty, Du_\infty, \mathbf{p}_\infty, D\mathbf{p}_\infty)|^q.
\end{aligned}$$

The integrand on the right hand side is a.e. convergent in Ω as $t \to \infty$ by condition A6 and since $\omega(t, x) \to \omega^*(x)$ for a.a. $x \in \Omega$. Further, it is integrable in Ω by conditions A2, A6 and estimate

$$\begin{aligned}
&|a_i(t, \cdot, \omega(t, \cdot), u_\infty, Du_\infty, \mathbf{p}_\infty, D\mathbf{p}_\infty) - a_{i,\infty}(\omega^*, u_\infty, Du_\infty, \mathbf{p}_\infty, D\mathbf{p}_\infty)|^q \leq \\
&\leq \text{const} \cdot \left(\|c_1(\omega)\|_{L^\infty(Q_\infty)} + \|c_1(\omega^*)\|_{L^\infty(Q_\infty)}\right) \left(|u_\infty|^p + |Du_\infty|^p + |\mathbf{p}_\infty|^p + |D\mathbf{p}_\infty|^p + \|k_1\|_{L^q(\Omega)}\right)
\end{aligned}$$

thus by Lebesgue's theorem we obtain $\|[\tilde{A}(\omega, u_\infty, \mathbf{p}_\infty; \omega, u, \mathbf{p})](t) - A_\infty(\omega, u_\infty, \mathbf{p}_\infty)\|_{V^*}^q \to 0$ as $t \to \infty$. The convergence of the last term in (32) can be proved similarly.

On the right hand side of (31) by Young's inequality we obtain

$$\begin{aligned}
&|\langle G(t) - G_\infty, u(t) - u_\infty\rangle + \langle F(t) - F_\infty, \mathbf{p}(t) - \mathbf{p}_\infty\rangle| \leq \\
&\frac{\varepsilon^p}{p}\|u(t) - u_\infty\|_V^p + \frac{\varepsilon^p}{p}\|\mathbf{p}(t) - \mathbf{p}_\infty\|_V^p + \frac{1}{q\varepsilon^q}\|G(t) - G_\infty\|_{V^*}^q + \frac{1}{q\varepsilon^q}\|F(t) - F_\infty\|_{V^*}^q
\end{aligned} \tag{33}$$

where the last two terms tend to 0 as $t \to \infty$.

Now, by choosing sufficiently small ε in (31) and by using (32), (33), the above convergences and the continuous embedding $L^p(\Omega) \hookrightarrow L^2(\Omega)$ we obtain

$$y'(t) + \text{const} \cdot y(t)^{\frac{p}{2}} + \text{const} \cdot \|\mathbf{p}(t) - \mathbf{p}_\infty\|_V^p \leq \varphi(t)$$

where $\varphi(t) \to 0$ as $t \to \infty$ and the constants are positive. It is not difficult to show that this inequality implies $\lim_{t\to\infty} y(t) = 0$ (see the proof of Theorem 2 in [14]), furthermore, by integrating (31) over $(t-1, t+1)$ one can deduce the convergences $\int_{t-1}^{t+1} \|u(s) - u_\infty\|_V^p \, ds \to 0$, $\int_{t-1}^{t+1} \|\mathbf{p}(s) - \mathbf{p}_\infty\|_V^p \, ds \to 0$, too. The proof of stabilization is complete. □

Consider the following functions for $i = 0, \dots, n$

$$a_i(t, x, \xi, \zeta_0, \zeta, \eta_0, \eta; w, v_1, v_2) = [\pi(w)](t, x)[\varphi(v_1)](t, x)[\psi(v_2)](t, x)P(\xi)\zeta_i|(\zeta_0, \zeta, \eta_0, \eta)|^{p-2}, \tag{34}$$

$$b_i(t, x, \xi, \zeta_0, \eta_0, \eta; w, v_1, v_2) = [\kappa(w)](t, x)[\lambda(v_1)](t, x)[\vartheta(v_2)](t, x)R(\xi)\eta_i|(\zeta_0, \eta_0, \eta)|^{p-2}. \tag{35}$$

Suppose

E3. a) Operators $\pi\colon L^\infty_{\text{loc}}(Q_\infty) \to L^\infty_{\text{loc}}(Q_\infty)$, $\varphi, \psi\colon L^p_{\text{loc}}(Q_\infty) \to L^\infty_{\text{loc}}(Q_\infty)$ are of Volterra type, further, for every $0 < T < \infty$, $\pi\colon L^\infty(Q_T) \to L^\infty(Q_T)$, $\varphi, \psi\colon L^p(Q_T) \to L^\infty(Q_T)$ are bounded, φ and ψ are continuous, and if (ω_k) is bounded in $L^\infty(Q_T)$ and $\omega_k \to \omega$ a.e. in Q_T then $\pi(\omega_k) \to \pi(\omega)$ in $L^\infty(Q_T)$. In addition, $P \in C(\mathbb{R})$, and there exists a positive lower bound for the values of π, φ, ψ, P.

b) There exist $\pi_\infty, \varphi_\infty, \psi_\infty \in L^\infty(\Omega)$ such that $\pi_\infty, \varphi_\infty, \psi_\infty \geq \text{const} > 0$, further, for every $w \in L^\infty(Q_\infty)$, $v_1 \in X^\infty \cap L^\infty(0, \infty; L^2(\Omega))$, $v_2 \in X^\infty \cap L^\infty(0, \infty; V)$,

$$\lim_{t\to\infty} \left(\|[\pi(w)](t, \cdot) - \pi_\infty\|_{L^\infty(\Omega)} + \|[\varphi(v_1)](t, \cdot) - \varphi_\infty\|_{L^\infty(\Omega)} + \|[\psi(v_2)](t, \cdot) - \psi_\infty\|_{L^\infty(\Omega)}\right) = 0.$$

E4. a) Operators $\kappa\colon L^\infty_{\rm loc}(Q_\infty) \to L^\infty_{\rm loc}(Q_\infty)$, $\lambda, \vartheta\colon L^p_{\rm loc}(Q_\infty) \to L^\infty_{\rm loc}(Q_\infty)$ are of Volterra type, further, for every $0 < T < \infty$, $\kappa\colon L^\infty(Q_T) \to L^\infty(Q_T)$, $\lambda, \vartheta\colon L^p(Q_T) \to L^\infty(Q_T)$ are bounded, λ and ϑ are continuous, and if (ω_k) is bounded in $L^\infty(Q_T)$ and $\omega_k \to \omega$ a.e. in Q_T then $\kappa(\omega_k) \to \kappa(\omega)$ in $L^\infty(Q_T)$. In addition, $R \in C(\mathbb{R})$, and there exists a positive lower bound for the values of π, φ, ψ, R.

b) There exist $\kappa_\infty, \lambda_\infty, \vartheta_\infty \in L^\infty(\Omega)$ such that $\kappa_\infty, \lambda_\infty, \vartheta_\infty \geq$ const > 0, further, for every $w \in L^\infty(Q_\infty), v_1 \in X^\infty \cap L^\infty(0,\infty; L^2(\Omega)), v_2 \in X^\infty \cap L^\infty(0,\infty; V)$

$$\lim_{t\to\infty} \left(\|[\kappa(w)](t,\cdot) - \kappa_\infty\|_{L^\infty(\Omega)} + \|[\lambda(v_1)](t,\cdot) - \lambda_\infty\|_{L^\infty(\Omega)} + \|[\vartheta(v_2)](t,\cdot) - \vartheta_\infty\|_{L^\infty(\Omega)} \right) = 0.$$

It is not difficult to prove (for some arguments see, e.g., [7, 3, 13])

Proposition 12. *Suppose $2 \leq p \leq 4$ and E3–E4, then the above (34)–(35) functions satisfy conditions A1–A3 A4*, A5–A6, B1–B3, B4*, B5–B6, AB with $p_1 = p_2 = p$.*

If we consider

$$a_i(t,x,\xi,\zeta_0,\zeta,\eta_0,\eta; w, v_1, v_2) = \zeta_i |(\zeta_0,\zeta)|^{p-2} + [\pi(w)](t,x)[\phi(v_1)](t,x) P(\xi) \zeta_i |(\zeta_0,\zeta,\eta_0,\eta)|^{r-2},$$
$$b_i(t,x,\xi,\zeta_0,\eta_0,\eta; w, v_1, v_2) = \zeta_i |(\eta_0,\eta)|^{p-2} + [\kappa(w)](t,x)(t,x)[\vartheta(v_2)](t,x) R(\xi) \eta_i |(\zeta_0,\eta_0,\eta)|^{r-2}$$

where $1 \leq r \leq 4$ and E3–E4 hold then it is easy to see that these functions satisfy conditions A1–A3, A4*, A5–A6, B1–B6, AB with $p_1 = p_2 = p \geq \max\{2, r\}$. E.g. operators π, φ may have the form

$$[\pi(w)](t,x) = \chi(t) \int_{Q_t} |w|^\beta + \pi_\infty(x), \quad [\varphi(v)](t,x) = \tilde\chi(t) \int_\Omega |d(t,x)||v(t,x)|^\beta dx + \varphi_\infty(x),$$

where $\lim_{t\to\infty} \chi(t) = 0$, $\lim_{t\to\infty} \tilde\chi(t) = 0$ and $d, \pi_\infty, \varphi_\infty \in L^\infty(\Omega), 1 \leq \alpha, 1 \leq \beta \leq 2$. The other operators may have similar form.

As an example for function f consider, e.g., $f(t,x,\xi,\zeta_0; v) = -(\xi - \omega^*(x)) \int_\Omega |v(t,x)|^\beta \, dx$ where $1 \leq \beta \leq 2$.

References

[1] R. A. Adams, *Sobolev spaces*, Academic Press, New York - San Francisco - London, 1975.

[2] J. Berkovits, V. Mustonen, Topological degree for perturbations of linear maximal monotone mappings and applications to a class of parabolic problems, *Rend. Mat. Ser. VII*, **12**, Roma (1992), 597-621.

[3] Á. Besenyei, On systems of parabolic functional differential equations, *Annales Univ. Sci. Budapest*, **47** (2004), 143-160.

[4] Á. Besenyei, Existence of weak solutions of a nonlinear system modelling fluid flow in porous media, *Electron J. Diff. Eqns.*, Vol. 2006(2006), No. 153, pp. 1-19., `http://ejde.math.txstate.edu/Volumes/2006/153/abstr.html`

[5] Á. Besenyei, On nonlinear parabolic variational inequalities containing nonlocal terms, to appear in *Acta Math. Hungar.*

[6] Á. Besenyei, Stabilization of solutions to a nonlinear system modelling fluid flow in porous media, submitted, preprint version: `http://www.cs.elte.hu/~badam/publications/stabil.pdf`

[7] Á. Besenyei, Examples for uniformly monotone operators arising in weak forms of elliptic problems, preprint version: `http://www.cs.elte.hu/~badam/publications/uniform.pdf`

[8] F. E. Browder, Pseudo-monotone operators and nonlinear elliptic boundary value problems on unbounded domains, *Proc. Natl. Acad. Sci. USA*, **74** (1977), 2659–2661.

[9] S. Cinca, Diffusion und Transport in porösen Medien bei veränderlichen Porosität, Diplomawork, University of Heidelberg, 2000.

[10] Yu. A. Dubinskiy, Nonlinear elliptic and parabolic equations (in Russian), in: Modern problems in mathematics, Vol. 9., Moscow, 1976.

[11] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Gauthier-Villars, Paris, 1969.

[12] J. D. Logan, M. R. Petersen, T. S. Shores, Numerical study of reaction-mineralogy-porosity changes in porous media, *Applied Mathematics and Computation*, **127** (2002), 149–164.

[13] L. Simon, On parabolic functional differential equations of general divergence form, *Proceedings of the Conference FSDONA 04*, Milovy, 2004, 280–291.

[14] L. Simon, On different types of nonlinear parabolic functional differential equations, *PU.M.A.*, **9** (1998), 181–192.

[15] E. Zeidler, *Nonlinear functional analysis and its applications II*, Springer, 1990.

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