Point M' belongs to the chord P_1Q_1', so by (1) we have

$$q_1' = \frac{p_1 - m'}{p_1 m' - 1}.$$

By (4), m and m' are the zeros of the quadratic $x^2 - 2e'x + 1$, so $mm' = 1$. Hence,

$$(p_1 - m')(p_1 - m) - (p_1 m - 1)(p_1 m' - 1) = (p_1^2 - 1)(1 - mm') = 0.$$

Thus,

$$q_1' = \frac{p_1 - m'}{p_1 m' - 1} = \frac{p_1 m - 1}{p_1 - m} = \frac{1}{q_1} = \overline{q_1}.$$

That is, q_1' is the conjugate of q_1, so Q_1' is the mirror image of Q_1 across the real axis EE'. Similar properties hold for Q_2', Q_3', and Q_4'. Consequently, the quadrilateral $Q_1'Q_2'Q_3'Q_4'$ is the mirror image of the rectangle $Q_1Q_2Q_3Q_4$.

Also solved by M. Bataille (France), R. Chapman (U. K.), R. J. Fisher, D. Fleischman, J.-P. Grivaux (France), O. P. Lossers (Netherlands), R. Stong, T. Wiandt, GCHQ Problem Solving Group (U. K.), and the proposer.

A Powerful Inequality

11752 [2014, 84]. *Proposed by Ádám Besenyei, Eötvös Loránd University, Budapest, Hungary.* Let $x_1, \dots, x_n$ be nonnegative numbers, where $n \geq 4$, and let $x_{n+1} = x_1$. For $p \geq 1$, prove that

$$\sum_{k=1}^{n} (x_k + x_{k+1})^p \leq \sum_{k=1}^{n} x_k^p + \left(\sum_{k=1}^{n} x_k \right)^p .$$

Solution by the proposer. Since the case $x_1 = \cdots = x_n = 0$ is clear, we may use homogeneity and assume $\sum_{j=1}^{n} x_j = 1$. Define $f\ [0, 1] \to \mathbb{R}$ by $f(w) = w - w^p$. Then for $p \geq 0$, we see that f is nonnegative and concave on [0, 1]. We must prove

$$\sum_{j=1}^{n} f(x_j) \leq \sum_{j=1}^{n} f(x_j + x_{j+1}).$$

Extending the subscripts periodically so that $x_{n+j} = x_j$ and writing y_j for $x_j + x_{j+1}$ yields

$$\begin{aligned} 2\sum_{j=1}^{n} f(x_j) - \sum_{j=1}^{n} f(y_j) &= \sum_{j=1}^{n} y_j^p \left[f(x_j/y_j) + f(x_{j+1}/y_j) \right] \\ &= \sum_{j=1}^{n} \left[y_j^p f(x_{j+1}/y_j) + y_{j+2}^p f(x_{j+2}/y_{j+2}) \right] \\ &\leq \sum_{j=1}^{n} \left[y_j f(x_{j+1}/y_j) + y_{j+2} f(x_{j+2}/y_{j+2}) \right]. \end{aligned}$$

Now $n \geq 4$, so $y_j + y_{j+2} \leq 1$, and then Jensen's inequality implies

$$\begin{aligned} & y_j f(x_{j+1}/y_j) + y_{j+2} f(x_{j+2}/y_{j+2}) \\ &= y_j f(x_{j+1}/y_j) + y_{j+2} f(x_{j+2}/y_{j+2}) + (1 - y_j - y_{j+2}) f(0) \leq f(y_{j+1}). \end{aligned}$$

Therefore,

$$2\sum_{j=1}^{n} f(x_j) \le \sum_{j=1}^{n} f(y_{j+1}) + \sum_{j=1}^{n} f(y_j) = 2\sum_{j=1}^{n} f(x_j + x_{j+1}),$$

as required.

Also solved by R. Stong.

An Integral Inequality with Many Zero Derivatives

11756 [2014, 170]. *Proposed by Paolo Perfetti, Department of Mathematics, University 'Tor Vergata', Rome, Italy.* Let f be a function from $[-1, 1]$ to $\mathbb{R}$ with continuous derivatives of all orders up to $2n + 2$. Given $f(0) = f''(0) = \cdots = f^{(2n)}(0) = 0$, prove

$$\frac{1}{2}((2n+2)!)^2(4n+5)\left(\int_{-1}^{1} f(x)\,dx\right)^2 \le \int_{-1}^{1} (f^{(2n+2)}(x))^2\,dx.$$

Solution by Eugene A. Herman, Grinnell College, Grinnell, IA. We use Taylor's formula with integral form of the remainder:

$$f(x) = \sum_{k=0}^{2n+1} \frac{x^k f^{(k)}(0)}{k!} + \int_0^x \frac{(x-t)^{2n+1} f^{(2n+2)}(t)}{(2n+1)!}\,dt.$$

The terms with k even are 0, and those with k odd integrate to 0 on the interval $[-1, 1]$, so

$$\begin{aligned}
\int_{-1}^{1} f(x)\,dx &= \int_{-1}^{1}\int_0^x \frac{(x-t)^{2n+1} f^{(2n+2)}(t)}{(2n+1)!}\,dt\,dx \\
&= -\int_{-1}^{0}\int_0^t \frac{(x-t)^{2n+1} f^{(2n+2)}(t)}{(2n+1)!}\,dx\,dt + \int_0^1\int_t^1 \frac{(x-t)^{2n+1} f^{(2n+2)}(t)}{(2n+1)!}\,dx\,dt \\
&= \frac{1}{(2n+2)!}\left(\int_{-1}^{0} (t+1)^{2n+2} f^{(2n+2)}(t)\,dt + \int_0^1 (t-1)^{2n+2} f^{(2n+2)}(t)\,dt\right).
\end{aligned}$$

Let $g(t) = t + 1$ for $t \in [-1, 0]$, $g(t) = t - 1$ for $t \in (0, 1]$. By the Cauchy–Schwarz inequality,

$$\begin{aligned}
((2n+2)!)^2\left(\int_{-1}^{1} f(x)\,dx\right)^2 &= \left(\int_{-1}^{1} g(t)^{2n+2} f^{(2n+2)}(t)\,dt\right)^2 \\
&\le \int_{-1}^{1} g(t)^{4n+4}\,dt \int_{-1}^{1} \left(f^{(2n+2)}(t)\right)^2 dt \\
&= \frac{2}{(4n+5)}\int_{-1}^{1} \left(f^{2n+2}(t)\right)^2 dt.
\end{aligned}$$

This completes the proof.

Also solved by U. Abel (Germany), R. Bagby, R. Boukharfane (Canada), R. Chapman (U. K.), P. P. Dályay (Hungary), O. Geupel (Germany), B. Karaivanov, O. Kouba (Syria), O. P. Lossers (Netherlands), J. Martínez (Spain), M. Omarjee (France), N. C. Singer, A. Stenger, R. Stong, R. Tauraso (Italy), and the proposer.