

# NOTES

Edited by **Sergei Tabachnikov**

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## Peano's Unnoticed Proof of Borel's Theorem

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**Abstract.** In 1876, P. du Bois-Reymond gave an example of an infinitely differentiable function whose Taylor series diverges everywhere except one point. By generalizing this example, G. Peano proved a theorem in 1884, often credited to É. Borel, which states that every power series is a Taylor series of some infinitely differentiable function. The aim of the paper is to recall Peano's unnoticed contributions to this result.

**1. INTRODUCTION.** One of the fundamental notions of real analysis is the Taylor expansion of infinitely differentiable functions. Everyone is familiar with the Taylor series of some common functions, and yet the general concept behind them is often misinterpreted. If  $f$  is an infinitely differentiable function at  $x = \alpha$ , then we are tempted to think, as Joseph-Louis Lagrange was, that  $f$  can be represented around  $\alpha$  by its (formal) Taylor series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(\alpha)}{k!} (x - \alpha)^k.$$

However, in general such a series need not converge in any neighborhood of  $\alpha$ , or if it does converge, its sum need not necessarily be  $f$ . The first such example (which became the standard textbook example) is due to Augustin-Louis Cauchy, who noted in 1823 [4, p. 230] that for the function defined as  $F(x) = \exp(-1/x^2)$  ( $x \neq 0$ ) and  $F(0) = 0$ , we have  $F^{(k)}(0) = 0$  for every  $k = 0, 1, \dots$  yielding a constant zero Taylor series that obviously does not equal to  $F$ . More than 50 years later, in 1876 [5, 6], the German mathematician Paul du Bois-Reymond gave the first example of an infinitely differentiable function whose Taylor series is divergent everywhere except at 0. Since then, numerous other examples were constructed illustrating the possible ill-behavior of Taylor series. In fact, Taylor series form a very diverse set inasmuch as it coincides with that of power series, as the following remarkable theorem states.

**Theorem.** *For any sequence of real numbers  $(c_n)$ , there is an infinitely differentiable function  $f: \mathbb{R} \rightarrow \mathbb{R}$  such that  $f^{(n)}(0) = c_n$  for all  $n = 0, 1, 2, \dots$*

This result was stated and proved in the doctoral thesis of the French mathematician Émile Borel (containing also the proof of the Heine–Borel theorem) published in 1895 [3, p. 35–44]. Therefore, it became known as Borel's theorem, and in the past decade several proofs were given (see [9, p. 55], [10], [11, p. 244], [12], [13, Ch. 19, Ex. 12], and also [1, p. 122], [2, p. 190, 193] for some other references). However, Borel's dissertation was not the first appearance of the theorem; the annotations written to

the famous analysis book of Angelo Genocchi and Giuseppe Peano published in 1884 [7] also included this result. After the publication of the book, it turned out that these annotations (in fact the whole book) were Peano's own contributions; see [8, p. 15–21] for details. In Annotation No. 67, Peano generalized the example of du Bois-Reymond by explicitly stating Borel's theorem and providing a concise constructive proof of it. In this short note, we recall Peano's unnoticed proof in a somewhat modernized form by filling in some details using some nice arguments due to du Bois-Reymond. Peano's proof, similarly to the aforementioned ones, is based only on termwise differentiability of uniformly convergent series of functions, and it is among the simplest ones; thus, it can also be directly included in a classroom setting.

**2. PEANO'S FUNCTION AND PROOF.** By following Peano, we consider the function

$$f(x) = \sum_{k=0}^{\infty} \frac{a_k x^k}{1 + b_k x^2}. \tag{1}$$

Let us suppose that  $(a_n)$  is an arbitrary and  $(b_n)$  is a positive sequence such that  $f$  is an infinitely many times termwise differentiable function. We then claim that

$$f(0) = a_0, \quad f'(0) = a_1, \quad \text{and} \tag{2}$$

$$\frac{f^{(n)}(0)}{n!} = a_n + \sum_{j=1}^{[n/2]} (-1)^j a_{n-2j} b_{n-2j}^j \quad (n \geq 2).$$

Indeed, the geometric series yields

$$\frac{a_k x^k}{1 + b_k x^2} = a_k x^k \sum_{j=0}^{\infty} (-1)^j b_k^j x^{2j} = \sum_{j=0}^{\infty} (-1)^j a_k b_k^j x^{2j+k} \quad (|b_k x^2| < 1).$$

Therefore,

$$\left( \frac{a_k x^k}{1 + b_k x^2} \right)^{(n)}(0) = \begin{cases} n! (-1)^j a_{n-2j} b_{n-2j}^j, & \text{if } k = n - 2j \text{ for some } j, \\ 0, & \text{otherwise.} \end{cases}$$

Now, from (2) it is clear that for any given  $(b_n)$  and  $(c_n)$ , we can uniquely determine  $(a_n)$  such that  $f^{(n)}(0) = c_n$  holds for  $n = 0, 1, \dots$ . Further, as Peano also noted, by choosing  $c_n = (n!)^2$  we obtain a function whose Taylor series around 0 is  $\sum_{n=0}^{\infty} n! x^n$ , which has radius of convergence 0 and thus diverges everywhere except at  $x = 0$ .

It remains to prove that the series (1) is infinitely many times termwise differentiable. This will be the case if  $a_n/b_n$  tends to 0 in a sufficiently high order such that  $f$  and the series obtained by arbitrary many formal termwise differentiations are uniformly convergent on bounded real intervals. More specifically, invoking du Bois-Reymond [5, 6], we show that

$$\left| \left( \frac{a_k x^k}{1 + b_k x^2} \right)^{(n)} \right| \leq (n + 1)! \frac{|a_k| k!}{b_k} |x|^{k-n-2} \quad (k \geq n + 2). \tag{3}$$

Then choosing, e.g.,  $b_k = (k!)^2|a_k|$ , we obtain

$$\sum_{k \geq n+2} \left| \left( \frac{a_k x^k}{1 + b_k x^2} \right)^{(n)} \right| \leq (n+1)! \sum_{k \geq n+2} \frac{|x|^{k-n-2}}{k!}, \quad (4)$$

where the right-hand side is uniformly convergent on every finite interval. Hence, the Weierstrass M-test finally yields the infinite termwise differentiability of  $f$ . To verify (3), let  $b > 0$  be arbitrary and consider

$$\frac{x^k}{b^2 + x^2} = \frac{x^{k-1}}{2} \left( \frac{1}{x + bi} + \frac{1}{x - bi} \right).$$

The generalized Leibniz rule implies

$$\begin{aligned} \left( \frac{x^k}{b^2 + x^2} \right)^{(n)} &= \frac{1}{2} \sum_{j=0}^n \binom{n}{j} (k-1) \cdots (k-1-n+j) x^{k-1-n+j} \\ &\quad \times \left( \frac{(-1)^j j!}{(x+bi)^{j+1}} + \frac{(-1)^j j!}{(x-bi)^{j+1}} \right) \\ &= \frac{1}{2} n! x^{k-n-2} \sum_{j=0}^n (-1)^j \frac{(k-1) \cdots (k-1-n+j)}{(n-j)!} \\ &\quad \times \left( \frac{x^{j+1}}{(x+bi)^{j+1}} + \frac{x^{j+1}}{(x-bi)^{j+1}} \right), \end{aligned}$$

where clearly

$$\left| \frac{x^{j+1}}{(x+bi)^{j+1}} + \frac{x^{j+1}}{(x-bi)^{j+1}} \right| \leq 2,$$

so that (3) follows.

So by choosing  $b_k = (k!)^2|a_k|$  (or any  $(b_k)$  such that the right-hand side of (4) is uniformly convergent on bounded intervals), for any given sequence  $(c_n)$  we can uniquely determine through (2) a sequence  $(a_n)$  such that the function (1) is infinitely differentiable and  $f^{(n)}(0) = c_n$  for  $n = 0, 1, \dots$ . This is Peano's theorem.

**3. REMARKS.** Peano also indicated the estimating (3) without explicitly mentioning the factors  $(n+1)!$  and  $k!$ , but referring to du Bois-Reymond. By using this estimate, du Bois-Reymond proved in [6] that the function

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k)!} \frac{x^{2k}}{x^2 + a_k^2}$$

is infinitely differentiable if  $a_k \rightarrow 0$ . Moreover, he showed that this function cannot be represented as a power series. However, his proof was not perfectly rigorous since he manipulated divergent series. Later, more general results were obtained; see [1] for a comprehensive history of infinitely differentiable functions that are not representable by power series (i.e., non-analytic functions).

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### The Deterioration of the Theorem

“Suppose  $\star$  is a binary operation on  $Z$ -plus, which satisfies the properties of Theorem 10.”

“Suppose  $m = 1$  and  $n - 1$  is prime.”

“Suppose further that  $x$ -sub- $r$   $\star$   $i$  equals  $x$ -sub- $s$   $\star$   $j$ .”

Suppose, suppose.

Like if only, if only.

If only we had one more if, we might be able to get to the then.

It's the bargaining stage of grief.

At this rate we'll never reach acceptance.

—Submitted by Marion Cohen