A brief history of the mean value theorem

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1 The theorems of Rolle, Lagrange and Cauchy
   The mean value theorem
   Rolle’s theorem
   Cauchy’s theorem

2 How to prove it?
   The classical proofs
   Peano’s theorem
   Application

3 Steps towards the modern form
   Rolle’s theorem
   Mean value theorem

4 Dispute between mathematicians
   Peano and Jordan
   Peano and Gilbert
The theorem in classical form

**Theorem (Mean value theorem)**

Let \( f : [a, b] \rightarrow \mathbb{R} \) be continuous and differentiable on \((a, b)\). Then there exists \( \xi \in (a, b) \) such that

\[
f'(\xi) = \frac{f(b) - f(a)}{b - a}.
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The theorem in classical form

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Geometrically:
The theorem in classical form

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- chord has slope \( \frac{f(b) - f(a)}{b - a} \)
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Theorems of Rolle, Lagrange and Cauchy

The mean value theorem

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Geometrically:
- chord has slope $\frac{f(b) - f(a)}{b - a}$
- there is a tangent line parallel to the chord

Physically:
- at some instant
  instantaneous velocity = average velocity
The theorem in classical form

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**Corollary (Mean value inequality)**

Let $f : [a, b] \to \mathbb{R}$ be continuous and differentiable on $(a, b)$. Then

$$\inf_{[a,b]} f' \leq \frac{f(b) - f(a)}{b - a} \leq \sup_{[a,b]} f'.$$
Theorems of Rolle, Lagrange and Cauchy

A special case: Rolle’s theorem

**Theorem (Rolle, 1690, for polynomials)**

Let $f : [a, b] \to \mathbb{R}$ be continuous and differentiable on $(a, b)$. If $f(a) = f(b)$, then there exists $\xi \in (a, b)$ such that $f'(\xi) = 0$. 
A special case: Rolle’s theorem

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Geometrically:
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Physically:
- if initial and terminal velocity are equal, then at some instant acceleration = 0
A generalization: Cauchy’s mean value theorem

**Theorem (Cauchy, 1823)**

Let $f, g : [a, b] \to \mathbb{R}$ be continuous and differentiable on $(a, b)$. Then there exists $\xi \in (a, b)$ such that

$$f'(\xi)(g(b) - g(a)) = g'(\xi)(f(b) - f(a)).$$
The theorems of Rolle, Lagrange and Cauchy

Cauchy’s theorem

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Let \( f, g : [a, b] \to \mathbb{R} \) be continuous and differentiable on \((a, b)\). Then there exists \( \xi \in (a, b) \) such that

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f'\left(\xi\right)(g(b) - g(a)) = g'\left(\xi\right)(f(b) - f(a)).
\]

Geometrically:

- curve \( \gamma(t) = (g(t), f(t)) \) \((t \in [a, b])\)
- direction of chord is \((g(b) - g(a), f(b) - f(a))\)
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- curve \( \gamma(t) = (g(t), f(t)) \) \( (t \in [a, b]) \)
- direction of chord is \( (g(b) - g(a), f(b) - f(a)) \)
- at some instant \( (g'(\xi), f'(\xi)) = 0 \) or tangent \( \parallel (g(b) - g(a), f(b) - f(a)) \)
The (nowadays) very straightforward proofs

Proof of Rolle’s theorem (U. Dini, 1878).

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Geometrically:

\[
\begin{align*}
  f(a) & = f(b) \\
  a & \leq \xi_1 \leq \xi_2 \leq b
\end{align*}
\]
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Proof of the mean value theorem (O. Bonnet).

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**Proof of the mean value theorem (O. Bonnet).**
- Let $h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b-a} (x-a)$,
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- thus $f'(\xi) - \frac{f(b) - f(a)}{b-a} = 0$. 
The (nowadays) very straightforward proofs

Proof of Cauchy’s theorem (O. Bonnet).

Geometrically:

\[
g(a) \quad f(a) \quad g(b) \quad f(b)
\]
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- We may assume \( g(a) \neq g(b) \), then let

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- thus \( f'(\xi) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(\xi) = 0. \)

Remark.

\[
\begin{align*}
  h(t) &= \frac{1}{g(a) - g(b)} \begin{vmatrix} f(t) & g(t) & 1 \\ f(a) & g(a) & 1 \\ f(b) & g(b) & 1 \end{vmatrix}.
\end{align*}
\]
A universal generalization: Peano’s mean value theorem

Theorem (Peano, 1884)

Let \( f, \varphi, \psi : [a, b] \to \mathbb{R} \) be continuous and differentiable on \((a, b)\). Then there exists \( x_1 \in (a, b) \) such that

\[ \begin{vmatrix}
  f'(x) & \varphi'(x) & \psi'(x) \\
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- then $h(a) = h(b)(= 0)$ (the determinant has identical rows),
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- then $h(a) = h(b) = 0$ (the determinant has identical rows),
- therefore by Rolle’s theorem $h' (\xi) = 0$ at some $\xi \in (a, b)$.
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\end{vmatrix} = 0.
\]

**Remark.**

- $\psi = 1$ is Cauchy’s theorem
- $\psi = 1, \varphi(x) = x$ is the mean value theorem
Who needs the mean value theorem, anyway?

**Theorem**

If $f : (a, b) \to \mathbb{R}$ is differentiable and $f' = 0$, then $f$ is constant.
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0 \neq \frac{f(c) - f(d)}{c - d}
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0 \neq \frac{f(c) - f(d)}{c - d} = f' (\xi) = 0.
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If \( f : (a, b) \rightarrow \mathbb{R} \) is differentiable and \( f' = 0 \), then \( f \) is constant.

**Theorem (L’Hospital’s rule, 1696, J. Bernoulli)**

If \( f, g : (a, b) \rightarrow \mathbb{R} \) are differentiable, \( g' \neq 0 \), \( \lim_{a^+} f = \lim_{a^+} g = 0 \) and \( \lim_{a^+} f'/g' \) exists, then

\[
\lim_{a^+} \frac{f}{g} = \lim_{a^+} \frac{f'}{g'}.
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$$

**Proof.**

$$
\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(<\xi)}{g'(<\xi)}
$$
Who needs the mean value theorem, anyway?

**Theorem**

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\]

**Proof.**

\[
\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(\xi)}{g'(\xi)}, \quad x \rightarrow a^+ \implies \xi \rightarrow a^+.
\]
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\]

**Remark.**

It is not difficult to avoid the mean value theorems (Bers, Cohen, Boas).
Historical development of Rolle’s theorem

Prehistory:
Historical development of Rolle’s theorem

Prehistory:

- Bashkara (1114–1185), Parameshvara (1380–1460)
  - astronomical calculations (inverse sine)
  - traces of Rolle’s and mean value theorem(?)
Historical development of Rolle’s theorem

- Michel Rolle (1652–1719)
Historical development of Rolle’s theorem

- Michel Rolle (1652–1719)

- Traité d’algèbre (1690)
  Démonstration d’une méthode pour résoudre les égalitez de tous les degrez (1691) (theorem found in 1910)

- $f'(x) = 0$ has at least one root between two consecutive roots of $f(x) = 0$
  (method of cascades for polynomials)
Historical development of Rolle’s theorem

- Michel Rolle (1691)
- Colin Maclaurin (1698–1746)
Historical development of Rolle’s theorem

- Michel Rolle (1691)
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- A second letter...concerning the roots of equations (1729)

\[
\text{Theorem III. In general, the Roots of the Equation } x^n - A x^{n-1} + B x^{n-2} - C x^{n-3} &c. = 0, \\
\text{are the Limits of the Roots of the Equation } n x^{n-1} \\
\quad - n - 1 \times A x^{n-2} + n - 2 \times B x^{n-3} &c. = 0, \\
\text{or of any Equation that is deduced from it by multiplying its Terms by any Arithmetical Progression } l \neq d, l \neq 2d, l \neq 3d &c. \text{ and conversely the Roots of this new Equation will be the Limits of the Roots of the proposed Equation } x^n - A x^{n-1} + B x^{n-2} &c. = 0.
\]
Historical development of Rolle’s theorem

- Michel Rolle (1691)
- Colin Maclaurin (1729)
- Leonhard Euler (1707–1783)
Historical development of Rolle’s theorem

- Michel Rolle (1691)
- Colin Maclaurin (1729)
- Leonhard Euler (1707–1783)

- Institutiones calculi differentialis (1755)
Historical development of Rolle’s theorem

- Michel Rolle (1691)
- Colin Maclaurin (1729)
- Leonhard Euler (1755)
- Joseph-Louis Lagrange (1736-1813)
Historical development of Rolle’s theorem

- Michel Rolle (1691)
- Colin Maclaurin (1729)
- Leonhard Euler (1755)
- Joseph-Louis Lagrange (1736-1813)

- Traité de la résolution des équations numériques de tous les degrés (1798)
Historical development of Rolle’s theorem

- Michel Rolle (1691)
- Colin Maclaurin (1729)
- Leonhard Euler (1755)
- Joseph-Louis Lagrange (1798)
- Moritz Wilhelm Drobisch (1802-1896)
Historical development of Rolle’s theorem

- Michel Rolle (1691)
- Colin Maclaurin (1729)
- Leonhard Euler (1755)
- Joseph-Louis Lagrange (1798)
- Moritz Wilhelm Drobisch (1802-1896)

- Grundzüge der Lehre von den höheren numerischen Gleichungen (1843)
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- Michel Rolle (1691)
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- Michel Rolle (1691)
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- Joseph-Louis Lagrange (1798)
- Moritz Wilhelm Drobisch (1843)
- Joseph Liouville (1809–1882)
Historical development of Rolle’s theorem

- Michel Rolle (1691)
- Colin Maclaurin (1729)
- Leonhard Euler (1755)
- Joseph-Louis Lagrange (1798)
- Moritz Wilhelm Drobisch (1843)
- Joseph Liouville (1809–1882)

- Extension du théorème de Rolle aux racines imaginaires des équations (1864)
Historical development of Rolle’s theorem

- Michel Rolle (1691)
- Colin Maclaurin (1729)
- Leonhard Euler (1755)
- Joseph-Louis Lagrange (1798)
- Moritz Wilhelm Drobisch (1843)
- Joseph Liouville (1864)
- Joseph-Alfred Serret (1819–1885)
The historical development of Rolle’s theorem:

- Michel Rolle (1691)
- Colin Maclaurin (1729)
- Leonhard Euler (1755)
- Joseph-Louis Lagrange (1798)
- Moritz Wilhelm Drobisch (1843)
- Joseph Liouville (1864)
- Joseph-Alfred Serret (1819–1885)

- *Cours d’algèbre supérieure (1866)*
- every continuous function is differentiable
Steps towards the modern form
Rolle’s theorem

Historical development of Rolle’s theorem

- Michel Rolle (1691)
- Colin Maclaurin (1729)
- Leonhard Euler (1755)
- Joseph-Louis Lagrange (1798)
- Moritz Wilhelm Drobisch (1843)
- Joseph Liouville (1864)
- Joseph-Alfred Serret (1819–1885)

- Cours d’algèbre supérieure (1866)

Théorème de Rolle.

123. La proposition connue sous le nom de théorème de Rolle est utile dans quelques circonstances, et elle se rattache directement à la théorie que nous exposons. Aussi croyons-nous devoir la présenter ici.

Théorème. — Si \( a \) et \( b \) désignent deux racines consécutives de l’équation \( f(x) = 0 \), en sorte que cette équation n’ait aucune autre racine comprise entre \( a \) et \( b \), l’équation \( f'(x) = 0 \), obtenue en égalant à zéro la dérivée de \( f(x) \), a au moins une racine comprise entre \( a \) et \( b \), et quand elle en a plusieurs, le nombre de ces racines est impair.
Historical development of Rolle’s theorem

- Michel Rolle (1691)
- Colin Maclaurin (1729)
- Leonhard Euler (1755)
- Joseph-Louis Lagrange (1798)
- Moritz Wilhelm Drobisch (1843)
- Joseph Liouville (1864)
- Joseph-Alfred Serret (1866)

and Rolle’s theorem became well-known...
Historical development of the mean value theorem

Prehistory:
Historical development of the mean value theorem

Prehistory:

- Bonaventura Cavalieri (1598-1647)
Steps towards the modern form
Mean value theorem

Historical development of the mean value theorem

Prehistory:

- Bonaventura Cavalieri (1598-1647)
- *Geometria indivisibilibus* (1635)
Historical development of the mean value theorem

Prehistory:

- Bonaventura Cavalieri
  - Geometria indivisibilibus (1635)

\[2 \text{ Ibid.: Si curva linea quaeunque tota sit in eodem plano, cui occurrat recta in duobus punctis, aut rectis lineis, vel in recta, & puncto, poterimus aliam rectam lineam praefatae aequidistantem ducere, quae tangat portionem curvae lineae inter duos predictos occursus continuatam.}\]

If a curved line is situated in one plane and if a straight line meets it in either two points, two line segments, or in a line segment and a point, then we can draw another straight line parallel to the previous line which touches the part of the curve situated between the two mentioned meetings.\[2\]
Historical development of the mean value theorem

- Joseph-Louis Lagrange (1736-1813)
Historical development of the mean value theorem

- Joseph-Louis Lagrange (1736-1813)

- Leçons sur le calcul des fonctions (1801)

\[
\text{Soit } D \text{ une quantité donnée qu'on pourra prendre aussi petite qu'on voudra; on pourra donc toujours donner à } i \text{ une valeur assez petite pour que la valeur de } V \text{ soit renfermée entre les limites } D \text{ et } -D; \text{ donc, puisqu'on a}
\]
\[
f(x+i) - f(x) = i[f'(x) + V],
\]
\[
il'ensuit que la quantité } f(x+i) - f(x) \text{ sera renfermée entre ces deux-ci}
\]
\[
i[f'(x) \pm D].
\]
Historical development of the mean value theorem

- Joseph-Louis Lagrange (1801)
- André-Marie Ampère (1775–1836)
Steps towards the modern form

Mean value theorem

Historical development of the mean value theorem

- Joseph-Louis Lagrange (1801)
- André-Marie Ampère (1775–1836)
  - Recherches sur quelques point de la théorie des fonctions dérivées (1806)
  - every function is differentiable (?)

\[ f(x + i) = f(x) + if'(x) + iI. \]
Historical development of the mean value theorem

- Joseph-Louis Lagrange (1801)
- André-Marie Ampère (1806)
- Augustin-Louis Cauchy (1789–1857)
Historical development of the mean value theorem

- Joseph-Louis Lagrange (1801)
- André-Marie Ampère (1806)
- Augustin-Louis Cauchy (1789–1857)

- Résumé des leçons sur le calcul infinitésimal (1823)
Historical development of the mean value theorem

- Joseph-Louis Lagrange (1801)
- André-Marie Ampère (1806)
- Augustin-Louis Cauchy (1789–1857)

- Résumé des leçons sur le calcul infinitésimal (1823)

Or il est aisé de voir que des raisonnements entièrement semblables à ceux dont nous avons fait usage pour démontrer l’équation précédente suffiront pour établir la formule

\[
\frac{f(X) - f(x_0)}{F(X) - F(x_0)} = \frac{f'[x_0 + \theta(X - x_0)]}{F'[x_0 + \theta(X - x_0)]},
\]

\(\theta\) désignant encore un nombre inférieur à l’unité, et \(F(x)\) une fonction nouvelle qui, toujours croissante ou décroissante depuis la limite \(x = x_0\), jusqu’à la limite \(x = X\), reste continue, avec sa dérivée \(F'(x)\), entre ces mêmes limites.
Historical development of the mean value theorem

- Joseph-Louis Lagrange (1801)
- André-Marie Ampère (1806)
- Augustin-Louis Cauchy (1823)
- Pierre-Ossian Bonnet (1819–1892)
Historical development of the mean value theorem

- Joseph-Louis Lagrange (1801)
- André-Marie Ampère (1806)
- Augustin-Louis Cauchy (1823)
- Pierre-Ossian Bonnet (1819–1892)

- Serret, Cours de calcul infinitésimal (1868)

\[
\text{Théorème I. — Soit } f(x) \text{ une fonction de } x \text{ qui reste continue pour les valeurs de } x \text{ comprises entre des limites données, et qui, pour ces valeurs, ait une dérivée } f'(x) \text{ déterminée. Si } x_0 \text{ et } X \text{ désignent deux valeurs de } x \text{ comprises entre les mêmes limites, on aura}
\]

\[
\frac{f(X) - f(x_0)}{X - x_0} = f'(x_1),
\]

\(x_1\) étant une valeur comprise entre \(x_0\) et \(X\).
Historical development of the mean value theorem

- Joseph-Louis Lagrange (1801)
- André-Marie Ampère (1806)
- Augustin-Louis Cauchy (1823)
- Pierre-Ossian Bonnet (1819–1892)

- Serret, Cours de calcul infinitésimal (1868)
Historical development of the mean value theorem

- Joseph-Louis Lagrange (1801)
- André-Marie Ampère (1806)
- Augustin-Louis Cauchy (1823)
- Pierre-Ossian Bonnet (1868)
- Ulisse Dini (1845–1918)
Historical development of the mean value theorem

- Joseph-Louis Lagrange (1801)
- André-Marie Ampère (1806)
- Augustin-Louis Cauchy (1823)
- Pierre-Ossian Bonnet (1868)
- Ulisse Dini (1845–1918)

  - Fondamenti per la teorica delle funzioni di variabili reali (1878)
  - perfect proof (and perfect book)
Historical development of the mean value theorem

- Joseph-Louis Lagrange (1801)
- André-Marie Ampère (1806)
- Augustin-Louis Cauchy (1823)
- Pierre-Ossian Bonnet (1868)
- Ulisse Dini (1878)
- Carl Gustav Axel Harnack (1851–1888)
Historical development of the mean value theorem

- Joseph-Louis Lagrange (1801)
- André-Marie Ampère (1806)
- Augustin-Louis Cauchy (1823)
- Pierre-Ossian Bonnet (1868)
- Ulisse Dini (1878)
- Carl Gustav Axel Harnack (1851–1888)
  
  - Die Elemente der Differential- und Integralrechnung (1881)
Historical development of the mean value theorem

- Joseph-Louis Lagrange (1801)
- André-Marie Ampère (1806)
- Augustin-Louis Cauchy (1823)
- Pierre-Ossian Bonnet (1868)
- Ulisse Dini (1878)
- Carl Gustav Axel Harnack (1851–1888)

- Die Elemente der Differential- und Integralrechnung (1881)
Historical development of the mean value theorem

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- Pierre-Ossian Bonnet (1868)
- Ulisse Dini (1878)
- Carl Gustav Axel Harnack (1881)
- Moritz Pasch (1843–1930)
Steps towards the modern form

Mean value theorem

Historical development of the mean value theorem

- Joseph-Louis Lagrange (1801)
- André-Marie Ampère (1806)
- Augustin-Louis Cauchy (1823)
- Pierre-Ossian Bonnet (1868)
- Ulisse Dini (1878)
- Carl Gustav Axel Harnack (1881)
- Moritz Pasch (1843–1930)
  - Einleitung in die Differential- und Integralrechnung (1882)
Historical development of the mean value theorem

- Joseph-Louis Lagrange (1801)
- André-Marie Ampère (1806)
- Augustin-Louis Cauchy (1823)
- Pierre-Ossian Bonnet (1868)
- Ulisse Dini (1878)
- Carl Gustav Axel Harnack (1881)
- Moritz Pasch (1882)

and the theorem seemed to be perfectly clear...
Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)
Dispute between mathematicians Peano and Jordan

Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)

  Cours d’analyse (1882) contains an erroneous proof
Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)

Cours d’analyse (1882) contains an erroneous proof of the mean value inequality.
Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)

Cours d’analyse (1882) contains an erroneous proof of the mean value inequality.

Proof of the mean value inequality.

Take $a = a_0 < a_2 < \cdots < a_n = b$. 
Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)

Cours d’analyse (1882) contains an erroneous proof

Proof of the mean value inequality.

Take \( a = a_0 < a_2 < \cdots < a_n = b \). Then

\[
    f(a_i) - f(a_{i-1}) = (a_i - a_{i-1})[f'(a_{i-1}) + \varepsilon_i].
\]
Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)

Cours d’analyse (1882) contains an erroneous proof

**Proof of the mean value inequality.**

Take \( a = a_0 < a_2 < \cdots < a_n = b \). Then

\[
f(a_i) - f(a_{i-1}) = (a_i - a_{i-1})[f'(a_{i-1}) + \varepsilon_i].
\]

Thus, with \( M = \max_{[a,b]} |f'|, m = \min_{[a,b]} |f'|, \varepsilon = \max_{i=1,\ldots,n} |\varepsilon_i| \), after summation

\[
(m - \varepsilon)(b - a) \leq f(b) - f(a) \leq (M + \varepsilon)(b - a).
\]
Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)

Cours d’analyse (1882) contains an erroneous proof

**Proof of the mean value inequality.**

Take \( a = a_0 < a_2 < \cdots < a_n = b \). Then

\[
f(a_i) - f(a_{i-1}) = (a_i - a_{i-1})[f'(a_{i-1}) + \varepsilon_i].
\]

Thus, with \( M = \max_{[a,b]} |f'|, m = \min_{[a,b]} |f'|, \varepsilon = \max_{i=1,\ldots,n} |\varepsilon_i| \), after summation

\[
(m - \varepsilon)(b - a) \leq f(b) - f(a) \leq (M + \varepsilon)(b - a).
\]

Let \( n \to \infty \), then \( \varepsilon \to 0 \).
Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)
- Giuseppe Peano (1858–1932)
Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)
- Giuseppe Peano (1858–1932)

Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)

- Giuseppe Peano (1858–1932)


**Problem.**

If \( f(x) = x^2 \sin \frac{1}{x} \), \( a_0 = 0 \), \( a_1 = 1/2n\pi \), \( a_2 = 1/(2n + 1)\pi \)

then \( f(a_1) = f(a_2) = 0 \), \( f'(a_1) = -1 \) and so \( \varepsilon_2 = 1 \).
Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)

- Giuseppe Peano (1858–1932)


Problem.

The continuity of the derivative is used implicitly in your proof. (Exactly the same error made in Jules Hoüel’s Cours de calcul infinitésimal (1879).)
Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)
- Giuseppe Peano (1858–1932)


**Problem.**

However, the theorem can be easily proved without assuming the continuity of the derivative.
Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)

Reply to Peano (1884).

- Giuseppe Peano (1858–1932)
Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)

  **Reply to Peano (1884).**

  I know that my proof requires the continuity of the derivative. I will correct the error in Volume 3 of my book.

- Giuseppe Peano (1858–1932)
Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)

Reply to Peano (1884).

However, I would be very happy if you send me a proof without that assumption, because I do not know any.

- Giuseppe Peano (1858–1932)
Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)

- Giuseppe Peano (1858–1932)

Reply to Jordan (1884).
Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)

- Giuseppe Peano (1858–1932)

Reply to Jordan (1884).

I learned it from my teacher Angelo Genocchi. It is due to O. Bonnet, see Serret (with imperfections) and Dini, Harnack, Pasch (perfectly).
Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)

Jordan’s book is modified accordingly (1887).

- Giuseppe Peano (1858–1932)
Correspondence of Peano and Gilbert

- Louis-Philippe Gilbert (1832–1892)
  Chair of analysis at Université Catholique de Louvain
  (and cousin of Charles de la Vallée Poussin)

- Giuseppe Peano (1858–1932)
Correspondence of Peano and Gilbert

- Louis-Philippe Gilbert (1832–1892)
  
  Reply to Peano (1884).

- Giuseppe Peano (1858–1932)
Correspondence of Peano and Gilbert

- Louis-Philippe Gilbert (1832–1892)

  **Reply to Peano (1884).**

  Without continuity of the derivative, the theorem is false.

- Giuseppe Peano (1858–1932)
Correspondence of Peano and Gilbert

- Louis-Philippe Gilbert (1832–1892)

Reply to Peano (1884).

If $f(x) = \sqrt{2x}$ ($x \in (0, a)$) and $f(x) = \sqrt{2(2a - x)}$ ($x \in (a, 2a)$), then $f(a + h) - f(a - h) = 0$ but there is no $\xi$ for which $f'(\xi) = 0$.

- Giuseppe Peano (1858–1932)
Correspondence of Peano and Gilbert

- Louis-Philippe Gilbert (1832–1892)

Reply to Peano (1884).

- Giuseppe Peano (1858–1932)
Correspondence of Peano and Gilbert

- Louis-Philippe Gilbert (1832–1892)

- Giuseppe Peano (1858–1932)

Reply to Gilbert (1884).
Correspondence of Peano and Gilbert

- Louis-Philippe Gilbert (1832–1892)
- Giuseppe Peano (1858–1932)

Reply to Gilbert (1884).

Your function is not differentiable at some point. Differentiability means equal left and right derivative.
Correspondence of Peano and Gilbert

- Louis-Philippe Gilbert (1832–1892)

- Giuseppe Peano (1858–1932)

Reply to Gilbert (1884).

The proof is easy, I learned it from my teacher A. Genocchi. It is due to Bonnet, see Serret (with imperfections) and Dini, Harnack, Pasch (perfectly).
Correspondence of Peano and Gilbert

- Louis-Philippe Gilbert (1832–1892)
- Giuseppe Peano (1858–1932)

Reply to Gilbert (1884).

And, without malice, I give you an exercise: show that if $f$ is differentiable then for all $\varepsilon > 0$ there is a partition $a = a_0 < \cdots < a_n = b$ such that

$$\left| \frac{f(a_{i+1}) - f(a_i)}{a_{i+1} - a_i} - f'(a_i) \right| < \varepsilon.$$
Correspondence of Peano and Gilbert

- Louis-Philippe Gilbert (1832–1892)
  
  Reply to Peano (1884).

- Giuseppe Peano (1858–1932)
Correspondence of Peano and Gilbert

- Louis-Philippe Gilbert (1832–1892)

Reply to Peano (1884).

I misunderstood your concept of discontinuity of derivative. Of course, I knew Bonnet’s proof.

- Giuseppe Peano (1858–1932)
Correspondence of Peano and Gilbert

- Louis-Philippe Gilbert (1832–1892)

Reply to Peano (1884).

Your exercise is not interesting for me at all, however, I provide you a solution.

- Giuseppe Peano (1858–1932)
References


Gallica, Bibliothèque nationale de France

Internet Archive (Digital Library)
The End

Thank you for your attention!
Drobisch’s proof of Rolle’s theorem

**Proof.**
Drobisch’s proof of Rolle’s theorem

Proof.

Let $f(x) = (x - \alpha_1) \cdots (x - \alpha_n) \phi(x)$ where $\alpha_1 < \cdots < \alpha_n$ and $\phi > 0$. 
Drobisch’s proof of Rolle’s theorem

Proof.

Let \( f(x) = (x - \alpha_1) \ldots (x - \alpha_n) \varphi(x) \) where \( \alpha_1 < \cdots < \alpha_n \) and \( \varphi > 0 \). Then

\[
\begin{align*}
    f'(\alpha_1) & = (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3) \cdots (\alpha_1 - \alpha_n)\varphi(\alpha_1), \\
    f'(\alpha_2) & = (\alpha_2 - \alpha_1)(\alpha_2 - \alpha_3) \cdots (\alpha_2 - \alpha_n)\varphi(\alpha_2), \\
    f'(\alpha_3) & = (\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2) \cdots (\alpha_3 - \alpha_n)\varphi(\alpha_3), \\
    & \vdots
\end{align*}
\]
Drobisch’s proof of Rolle’s theorem

Proof.

Let \( f(x) = (x - \alpha_1) \cdots (x - \alpha_n) \varphi(x) \) where \( \alpha_1 < \cdots < \alpha_n \) and \( \varphi > 0 \).

Then

\[
\begin{align*}
  f'(\alpha_1) &= (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3) \cdots (\alpha_1 - \alpha_n) \varphi(\alpha_1), \\
  f'(\alpha_2) &= (\alpha_2 - \alpha_1)(\alpha_2 - \alpha_3) \cdots (\alpha_2 - \alpha_n) \varphi(\alpha_2), \\
  f'(\alpha_3) &= (\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2) \cdots (\alpha_3 - \alpha_n) \varphi(\alpha_3), \\
  &\vdots
\end{align*}
\]

Therefore, \( f'(\alpha_i) \) and \( f'(\alpha_{i+1}) \) have different signs.
Drobisch’s proof of Rolle’s theorem

Proof.

Let \( f(x) = (x - \alpha_1) \cdots (x - \alpha_n) \varphi(x) \) where \( \alpha_1 < \cdots < \alpha_n \) and \( \varphi > 0 \). Then

\[
\begin{align*}
f'(\alpha_1) &= (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3) \cdots (\alpha_1 - \alpha_n) \varphi(\alpha_1), \\
f'(\alpha_2) &= (\alpha_2 - \alpha_1)(\alpha_2 - \alpha_3) \cdots (\alpha_2 - \alpha_n) \varphi(\alpha_2), \\
f'(\alpha_3) &= (\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2) \cdots (\alpha_3 - \alpha_n) \varphi(\alpha_3), \\
&\vdots
\end{align*}
\]

Therefore, \( f'(\alpha_i) \) and \( f'(\alpha_{i+1}) \) have different signs thus \( f' \) has a root between \( \alpha_i \) and \( \alpha_{i+1} \).
Serret’s proof of Rolle’s theorem

Proof.
Proof.

Let $f(a) = f(b) = 0$ and $f > 0$ on $(a, b)$. 
Serret’s proof of Rolle’s theorem

**Proof.**

Let \( f(a) = f(b) = 0 \) and \( f > 0 \) on \((a, b)\). Since

\[
\frac{f(a + h)}{f'(a + h)} = \frac{hf'(a) + \frac{1}{2} h^2 f''(\xi)}{f'(a) + hf''(\eta)} = h \frac{1 + \text{small}}{1 + \text{small}},
\]
Serret’s proof of Rolle’s theorem

Proof.

Let \( f(a) = f(b) = 0 \) and \( f > 0 \) on \((a, b)\). Since

\[
\frac{f(a + h)}{f'(a + h)} = \frac{hf'(a) + \frac{1}{2}h^2f''(\xi)}{f'(a) + hf''(\eta)} = h \frac{1 + \text{small}}{1 + \text{small}},
\]

thus \( \frac{f(a+h)}{f'(a+h)} > 0 \) for small \( h \) and so \( f'(a + h) > 0 \).
Serret’s proof of Rolle’s theorem

Proof.

Let \( f(a) = f(b) = 0 \) and \( f > 0 \) on \((a, b)\). Since

\[
\frac{f(a + h)}{f'(a + h)} = \frac{hf'(a) + \frac{1}{2} h^2 f''(\xi)}{f'(a) + hf''(\eta)} = h \frac{1 + \text{small}}{1 + \text{small}},
\]

thus \( \frac{f(a + h)}{f'(a + h)} > 0 \) for small \( h \) and so \( f'(a + h) > 0 \). Similarly, \( f'(b - h) < 0 \) for small \( h \).
Serret’s proof of Rolle’s theorem

Proof.
Let \( f(a) = f(b) = 0 \) and \( f > 0 \) on \((a, b)\). Since

\[
\frac{f(a + h)}{f'(a + h)} = \frac{hf'(a) + \frac{1}{2}h^2f''(\xi)}{f'(a) + hf''(\eta)} = h\frac{1 + \text{small}}{1 + \text{small}},
\]

thus \( \frac{f(a+h)}{f'(a+h)} > 0 \) for small \( h \) and so \( f'(a + h) > 0 \). Similarly, \( f'(b - h) < 0 \) for small \( h \). Therefore, \( f' \) has a root between \( a + h \) and \( b - h \). \qed
Cauchy’s proof of the mean value inequality

Proof.
Cauchy’s proof of the mean value inequality

Proof.

Let $\varepsilon > 0$ and $\delta > 0$ be such that if $0 < i < \delta$ ($i=$indeterminate), then

$$f'(x) - \varepsilon < \frac{f(x + i) - f(x)}{i} < f'(x) + \varepsilon.$$
Cauchy’s proof of the mean value inequality

Proof.

Let \( \varepsilon > 0 \) and \( \delta > 0 \) be such that if \( 0 < i < \delta \) (\( i \) = indeterminate), then

\[
f'(x) - \varepsilon < \frac{f(x + i) - f(x)}{i} < f'(x) + \varepsilon.
\]

Take \( a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b \) such that \( x_i - x_{i-1} < \delta \).
Cauchy’s proof of the mean value inequality

Proof.

Let $\varepsilon > 0$ and $\delta > 0$ be such that if $0 < i < \delta$ ($i=$indeterminate), then

$$f'(x) - \varepsilon < \frac{f(x + i) - f(x)}{i} < f'(x) + \varepsilon.$$ 

Take $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ such that $x_i - x_{i-1} < \delta$, then

$$\min_{[a,b]} f' - \varepsilon \leq f'(x_i) - \varepsilon < \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} < f'(x_i) + \varepsilon \leq \max_{[a,b]} f' + \varepsilon,$$
Cauchy’s proof of the mean value inequality

Proof.

Let \( \varepsilon > 0 \) and \( \delta > 0 \) be such that if \( 0 < i < \delta \) (\( i \) indeterminate), then

\[
f'(x) - \varepsilon < \frac{f(x + i) - f(x)}{i} < f'(x) + \varepsilon.
\]

Take \( a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b \) such that \( x_i - x_{i-1} < \delta \), then

\[
\min_{[a,b]} f' - \varepsilon \leq f'(x_i) - \varepsilon < \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} < f'(x_i) + \varepsilon \leq \max_{[a,b]} f' + \varepsilon,
\]

and so by averaging with weights \( (x_i - x_{i-1}) \) one obtains

\[
\min_{[a,b]} f' - \varepsilon < \frac{f(b) - f(a)}{b - a} < \max_{[a,b]} f' + \varepsilon.
\]
Cauchy’s proof of the mean value inequality

Proof.

Let \( \varepsilon > 0 \) and \( \delta > 0 \) be such that if \( 0 < i < \delta \) \((i=\text{indeterminate})\), then

\[
f'(x) - \varepsilon < \frac{f(x + i) - f(x)}{i} < f'(x) + \varepsilon.
\]

Take \( a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b \) such that \( x_i - x_{i-1} < \delta \), then

\[
\min_{[a,b]} f' - \varepsilon \leq f'(x_i) - \varepsilon < \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} < f'(x_i) + \varepsilon \leq \max_{[a,b]} f' + \varepsilon,
\]

and so by averaging with weights \((x_i - x_{i-1})\) one obtains

\[
\min_{[a,b]} f' - \varepsilon < f'(b) - f(a) < \max_{[a,b]} f' + \varepsilon.
\]

The assertion follows by \( \varepsilon \to 0 \).
Cauchy’s proof of the mean value inequality

Remark.

- Problem: $\delta$ depends on $x$ (one needs the continuity of the derivative, as later noted Peano).
- The notion of mean (moyenne) was introduced by Cauchy in Analyse algébrique (1821).
- Cauchy proved rigorously the intermediate value theorem in Analyse algébrique (1821) so the mean value thereom also follows.
Cauchy’s proof of the generalized mean value theorem

Proof.
Cauchy’s proof of the generalized mean value theorem

Proof.

Assume $g' > 0$ and let $A := \min_{[a,b]} \frac{f'}{g'}$, $B := \max_{[a,b]} \frac{f'}{g'}$. 

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Cauchy’s proof of the generalized mean value theorem

Proof.

Assume $g' > 0$ and let $A := \min_{[a,b]} f'/g'$, $B := \max_{[a,b]} f'/g'$. Then

$$f'(x) - Ag'(x) \geq 0 \quad \text{and} \quadBg'(x) - f'(x) \geq 0 \quad (x \in (a, b)).$$
Cauchy’s proof of the generalized mean value theorem

Proof.

Assume \( g' > 0 \) and let \( A := \min_{[a,b]} \frac{f'}{g'} \), \( B := \max_{[a,b]} \frac{f'}{g'} \). Then

\[
f'(x) - Ag'(x) \geq 0 \quad \text{and} \quad Bg'(x) - f'(x) \geq 0 \quad (x \in (a, b)).
\]

Therefore \((f - Ag)\) and \((Bg - f)\) are increasing.
Cauchy’s proof of the generalized mean value theorem

Proof.

Assume \( g' > 0 \) and let \( A := \min_{[a,b]} \frac{f'}{g'} \), \( B := \max_{[a,b]} \frac{f'}{g'} \). Then

\[
\frac{d}{dx} f(x) - A g'(x) \geq 0 \quad \text{and} \quad B g'(x) - \frac{d}{dx} f(x) \geq 0 \quad (x \in (a, b)).
\]

Therefore \((f - Ag)\) and \((Bg - f)\) are increasing thus

\[
f(b) - f(a) - A(g(b) - g(a)) \geq 0 \quad \text{and} \quad B(g(b) - g(a)) - (f(b) - f(a)) \geq 0.
\]
Cauchy’s proof of the generalized mean value theorem

Proof.

Assume \( g' > 0 \) and let \( A := \min_{[a,b]} \frac{f'}{g'} \), \( B := \max_{[a,b]} \frac{f'}{g'} \). Then

\[
f'(x) - Ag'(x) \geq 0 \quad \text{and} \quad Bg'(x) - f'(x) \geq 0 \quad (x \in (a, b)).
\]

Therefore \((f - Ag)\) and \((Bg - f)\) are increasing thus

\[
f(b) - f(a) - A(g(b) - g(a)) \geq 0 \quad \text{and} \quad B(g(b) - g(a)) - (f(b) - f(a)) \geq 0.
\]

So

\[
A(g(b) - g(a)) \leq f(b) - f(a) \leq B(g(b) - g(a)),
\]

and

\[
A \leq \frac{f(b) - f(a)}{g(b) - g(a)} \leq B
\]

which is Cauchy’s mean value inequality.
Cauchy’s proof of the generalized mean value theorem

Proof.

Assume $g' > 0$ and let $A := \min_{[a,b]} f'/g'$, $B := \max_{[a,b]} f'/g'$. Then

$$f'(x) - Ag'(x) \geq 0 \quad \text{and} \quad Bg'(x) - f'(x) \geq 0 \quad (x \in (a,b)).$$

Therefore $(f - Ag)$ and $(Bg - f)$ are increasing thus

$$f(b) - f(a) - A(g(b) - g(a)) \geq 0 \quad \text{and} \quad B(g(b) - g(a)) - (f(b) - f(a)) \geq 0.$$

So

$$A(g(b) - g(a)) \leq f(b) - f(a) \leq B(g(b) - g(a)),$$

and

$$A \leq \frac{f(b) - f(a)}{g(b) - g(a)} \leq B$$

which is Cauchy’s mean value inequality. Cauchy’s mean value theorem follows by the intermediate value theorem and the continuity of $f'/g'$. □
Bonnet’s proof of the mean value theorem

Proof.
Bonnet’s proof of the mean value theorem

Proof.

Denote $A = \frac{f(b) - f(a)}{b-a}$ and $\varphi(x) = [f(x) - Ax] - [f(a) - Aa]$. 
Bonnet’s proof of the mean value theorem

Proof.

Denote \( A = \frac{f(b) - f(a)}{b - a} \) and \( \varphi(x) = [f(x) - Ax] - [f(a) - Aa] \). Then \( \varphi(a) = \varphi(b) = 0 \). Therefore, if \( \varphi \) is not constant, it has an extremum at some \( x_1 \in (a, b) \) and there \( \varphi'(x_1) = 0 \). Otherwise, \( \varphi' = 0 \).
Bonnet’s proof of the mean value theorem

Proof.

Denote \( A = \frac{f(b) - f(a)}{b-a} \) and \( \varphi(x) = [f(x) - Ax] - [f(a) - Aa] \). Then \( \varphi(a) = \varphi(b) = 0 \). Therefore, if \( \varphi \) is not constant, it has an extremum at some \( x_1 \in (a, b) \) and there \( \varphi'(x_1) = 0 \). Otherwise, \( \varphi' = 0 \).

Remark.

• No reference to Rolle’s theorem, but a complete proof.

• minor “imperfection” (as criticized by Peano):
  “if \( \varphi \) is not constant, then it should increase or decrease at \( a \)” is not true, for instance, \( x^2 \sin \frac{1}{x} \) at \( x = 0 \).

• Analogous proof for Cauchy’s mean value theorem: \( A = \frac{f(b) - f(a)}{g(b) - g(a)} \) and 
  \( \varphi(x) = [f(x) - Ag(x)] - [f(a) - Ag(a)] \).