A brief history of the mean value theorem

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Outline

1. The theorems of Rolle, Lagrange and Cauchy
   - The mean value theorem
   - Rolle's theorem
   - Cauchy's theorem
2. How to prove it?
   - The classical proofs
   - Peano's theorem
   - Application
3. Steps towards the modern form
   - Rolle's theorem
   - Mean value theorem
4. Dispute between mathematicians
   - Peano and Jordan
   - Peano and Gilbert

The theorem in classical form

Theorem (Mean value theorem)

Let $f : [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b). Then there exists $\xi \in (a, b)$ such that

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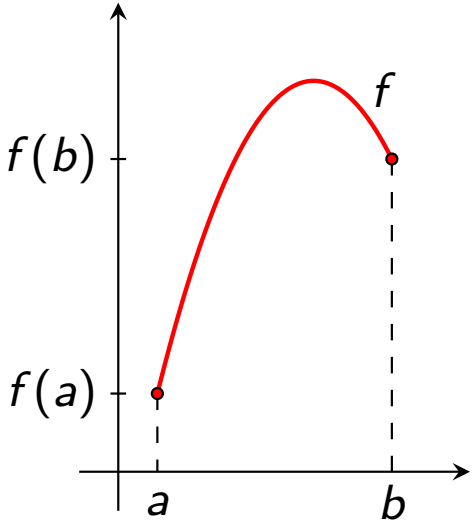

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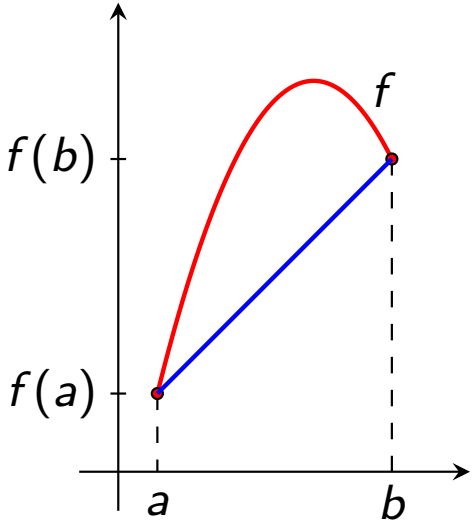

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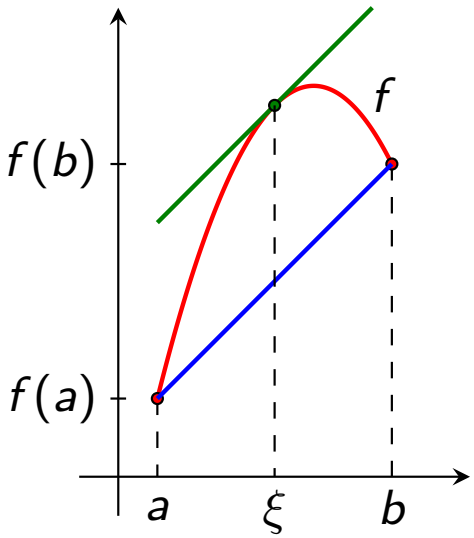

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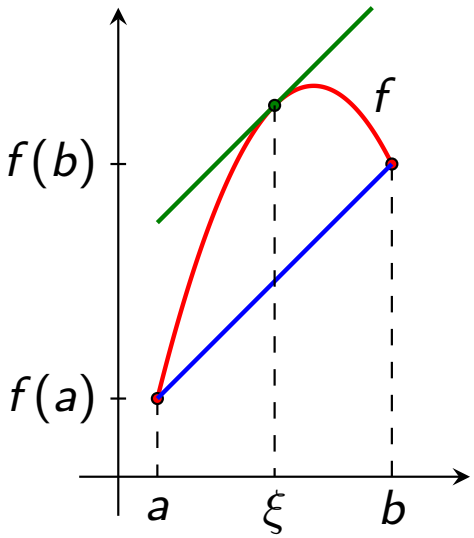

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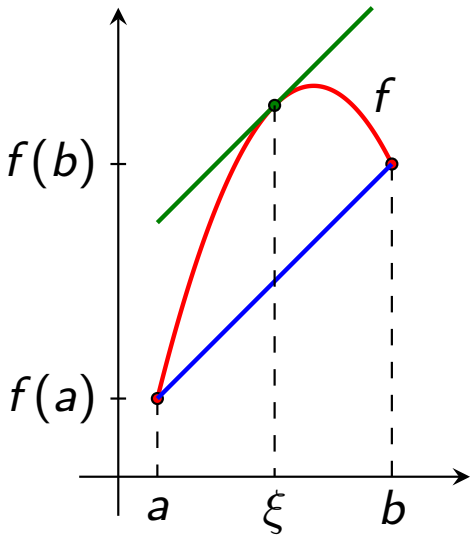

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Physically:

- at some instant
  instantaneous velocity = average velocity

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Corollary (Mean value inequality)

Let $f : [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b). Then

$$\inf_{[a,b]} f' \leq \frac{f(b) - f(a)}{b - a} \leq \sup_{[a,b]} f'.$$

A special case: Rolle's theorem

Theorem (Rolle, 1690, for polynomials)

Let $f : [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b). If $f(a) = f(b)$, then there exists $\xi \in (a, b)$ such that $f'(\xi) = 0$.

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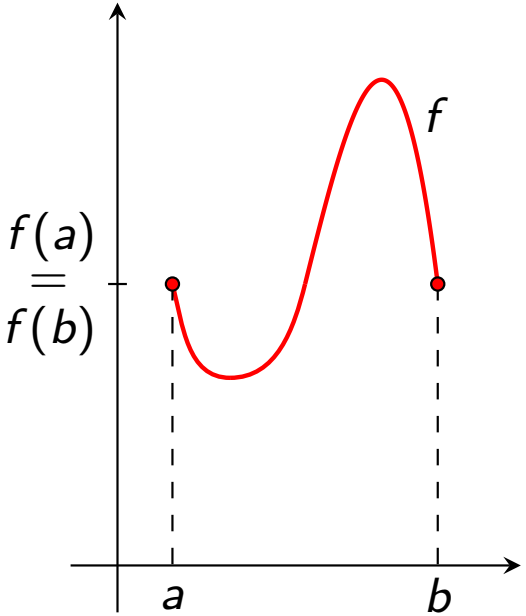

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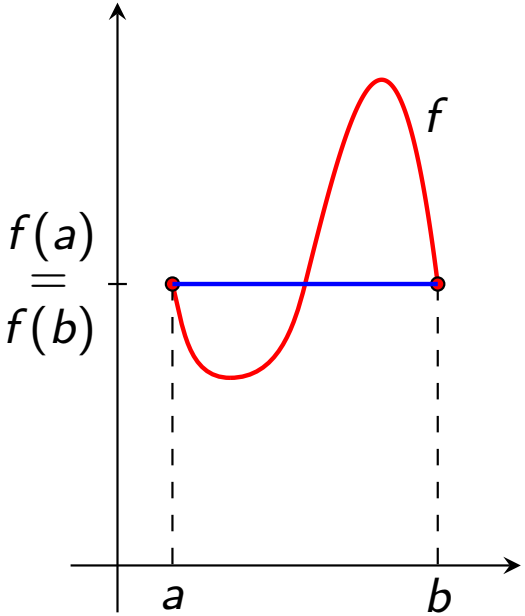

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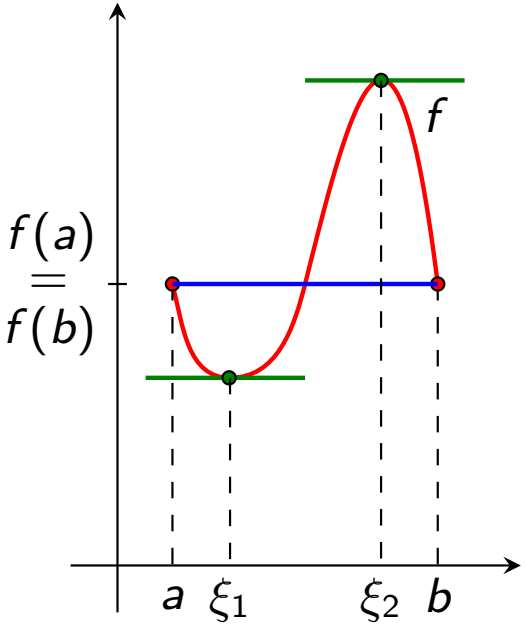

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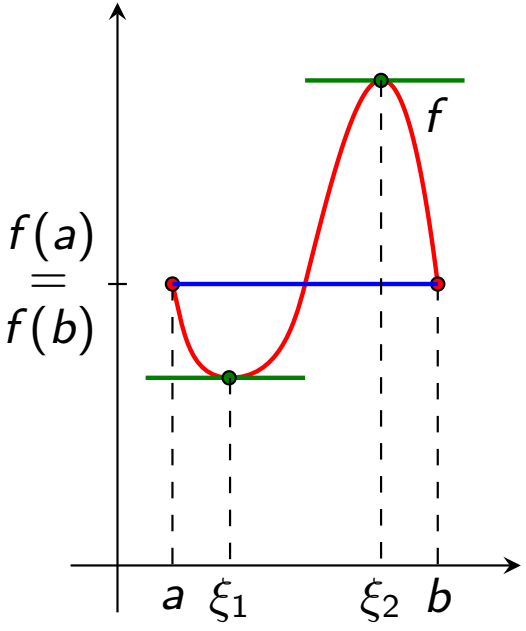

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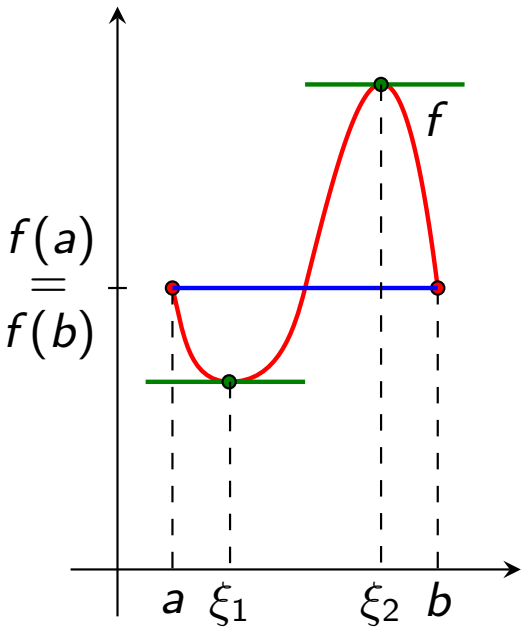

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Physically:

- if initial and terminal velocity are equal, then at some instant acceleration $= 0$

A generalization: Cauchy's mean value theorem

Theorem (Cauchy, 1823)

Let $f, g\colon [a,b] \to \mathbb{R}$ be continuous and differentiable on (a,b). Then there exists $\xi \in (a,b)$ such that

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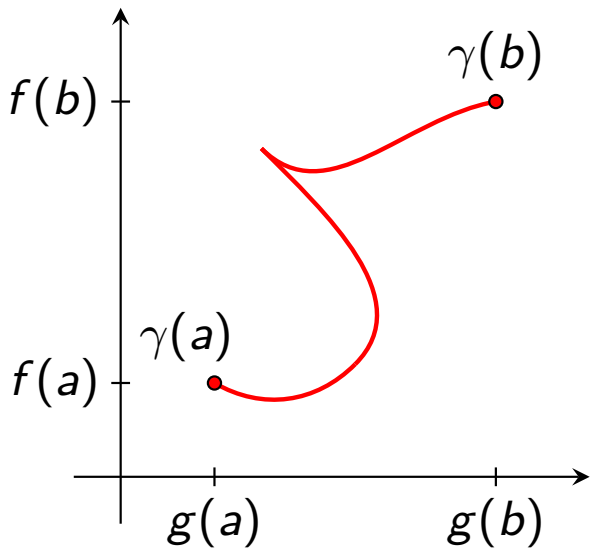

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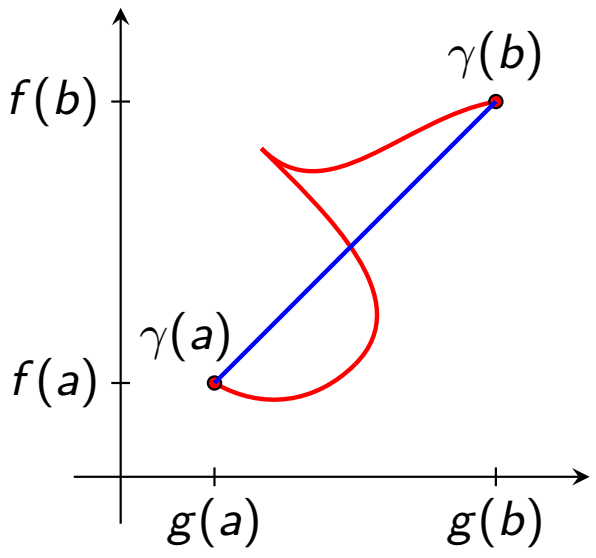

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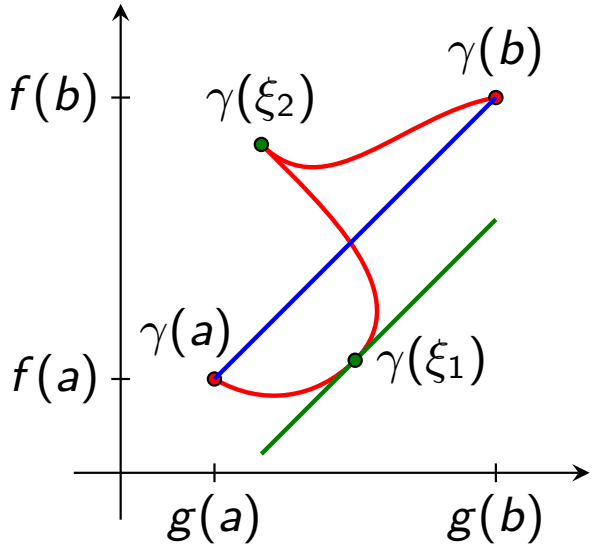

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- direction of chord is $(g(b)-g(a), f(b)-f(a))$
- at some instant $(g'(\xi), f'(\xi)) = 0$ or tangent $\parallel (g(b)-g(a), f(b)-f(a))$

The (nowadays) very straightforward proofs

Proof of Rolle's theorem (U. Dini, 1878).

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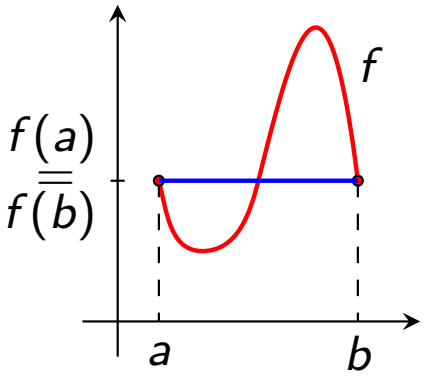

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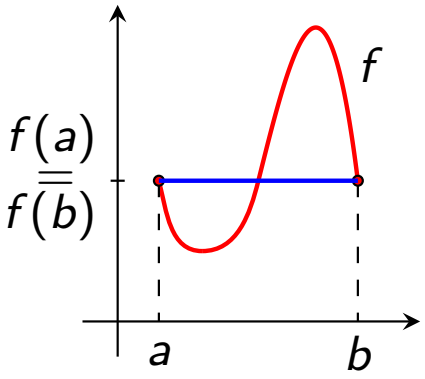

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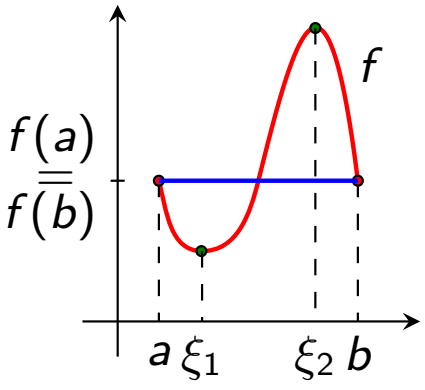

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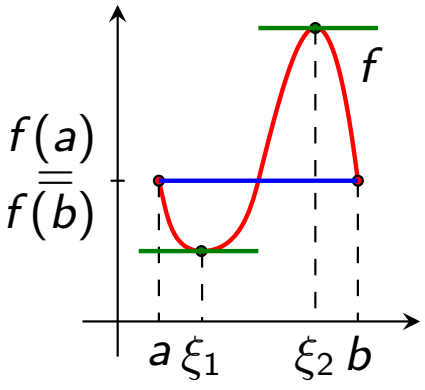

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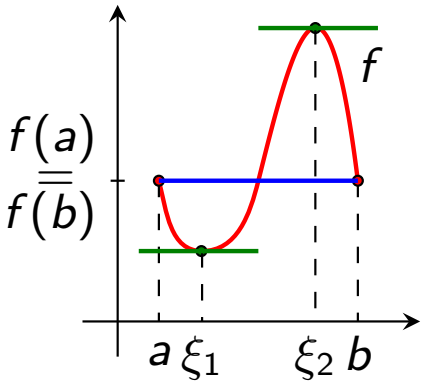

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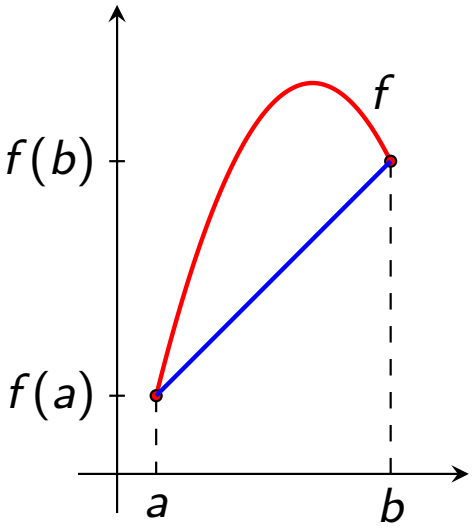

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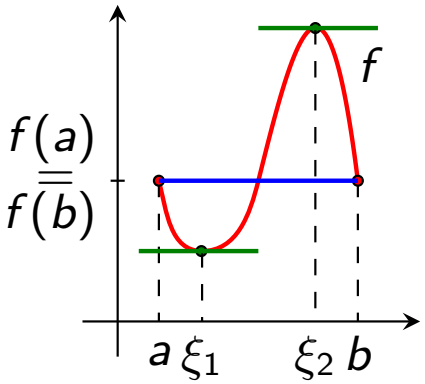

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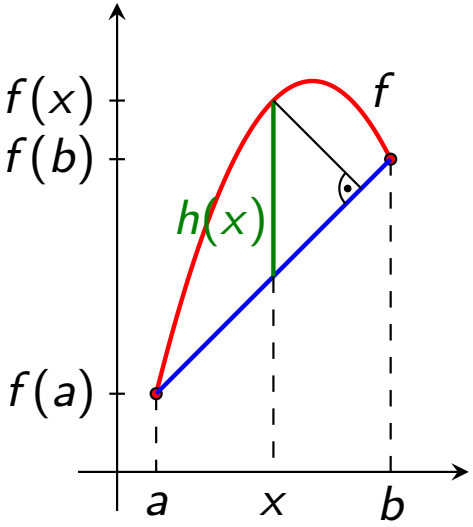

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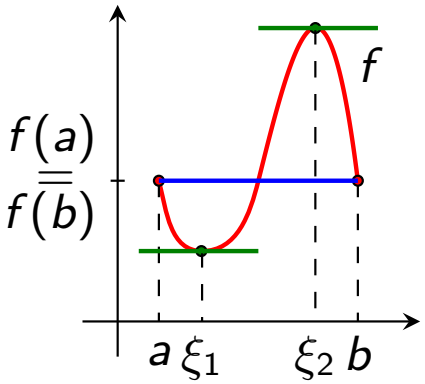

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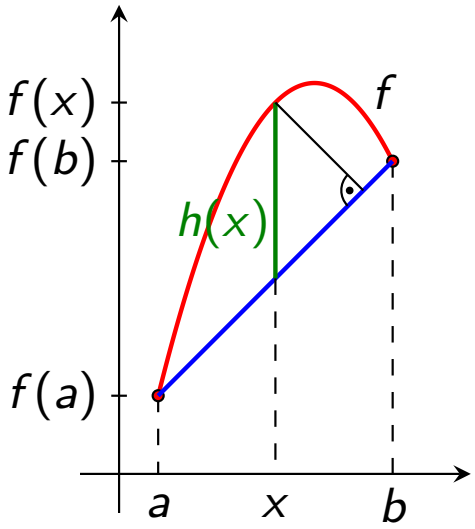

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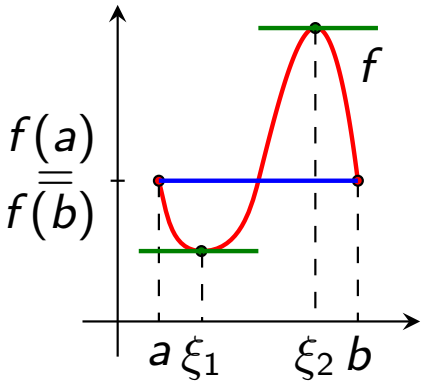

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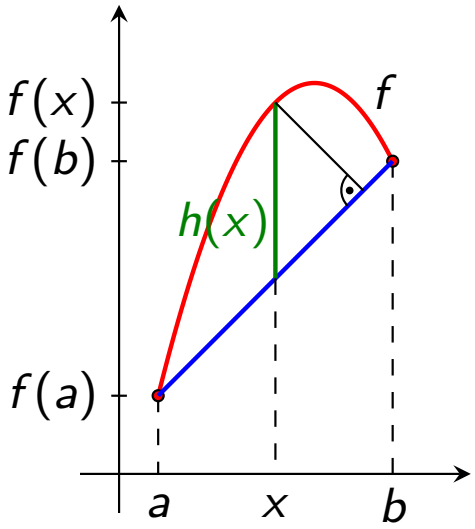

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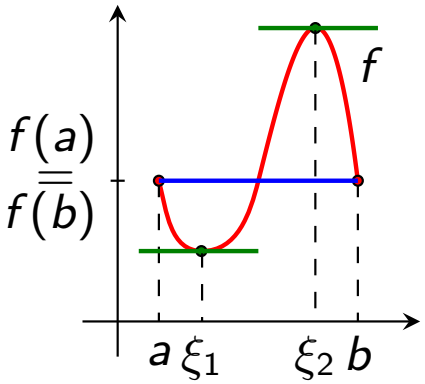

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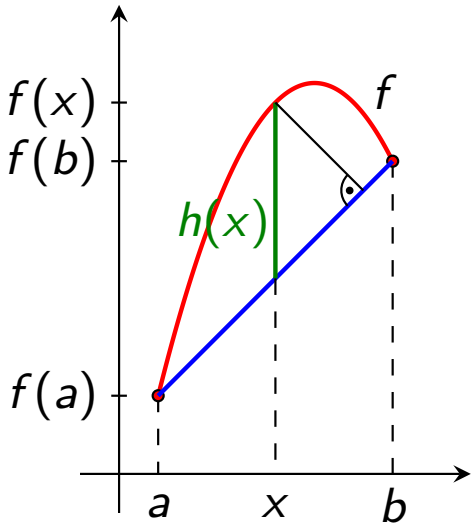

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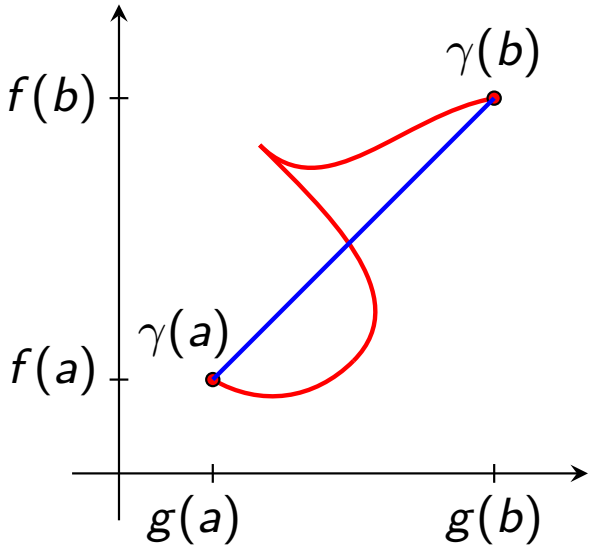

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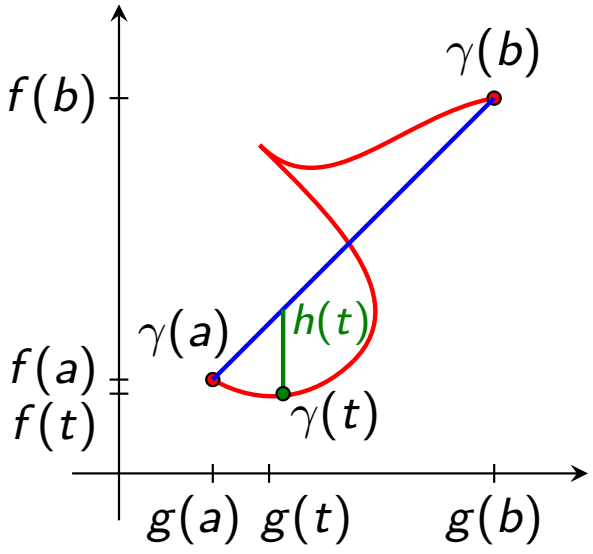

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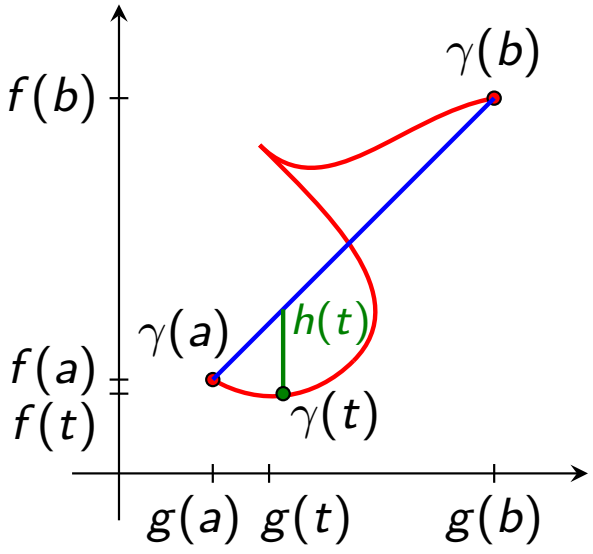

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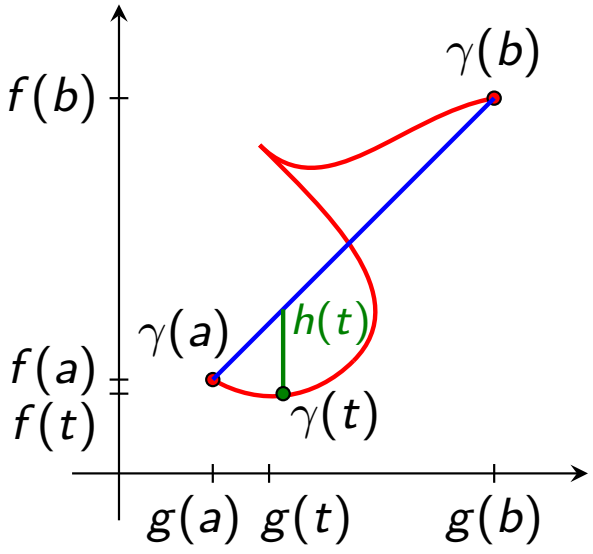

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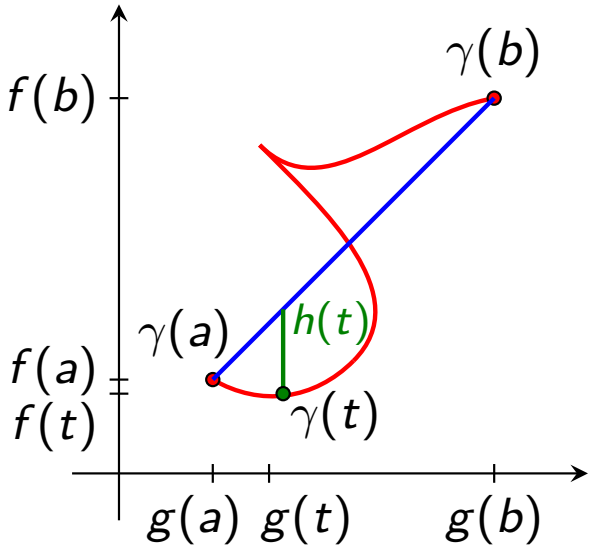

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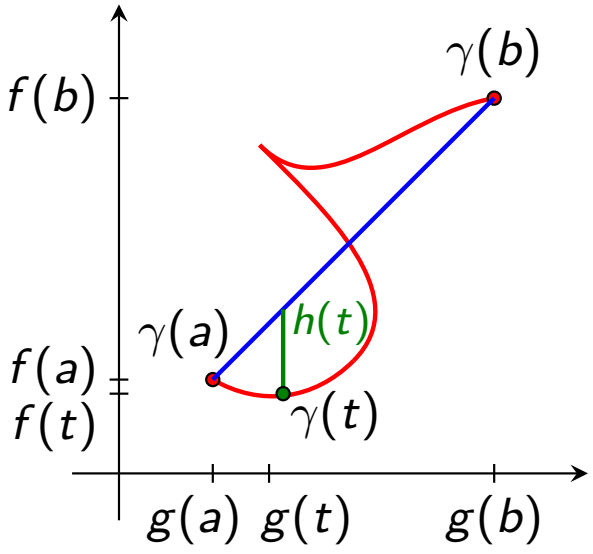
Geometrically:



Remark.

$$h(t) = \frac{1}{g(a) - g(b)} \begin{vmatrix} f(t) & g(t) & 1 \\ f(a) & g(a) & 1 \\ f(b) & g(b) & 1 \end{vmatrix}.$$

A universal generalization: Peano's mean value theorem

Theorem (Peano, 1884)

Let $f, \varphi, \psi \colon [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b). Then there exists $x_1 \in (a, b)$ such that

$$\begin{vmatrix} f'(x_1) & \varphi'(x_1) & \psi'(x_1) \\ f(a) & \varphi(a) & \psi(a) \\ f(b) & \varphi(b) & \psi(b) \end{vmatrix} = 0.$$

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Proof of Peano's theorem.

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- then $h(a) = h(b) (= 0)$ (the determinant has identical rows),

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- then $h(a) = h(b) (= 0)$ (the determinant has identical rows),
- therefore by Rolle's theorem $h'(\xi) = 0$ at some $\xi \in (a, b)$. $\square$

A universal generalization: Peano's mean value theorem

Theorem (Peano, 1884)

Let $f, \varphi, \psi \colon [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b). Then there exists $x_1 \in (a, b)$ such that

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Remark.

- $\psi = 1$ is Cauchy's theorem
- $\psi = 1$, $\varphi(x) = x$ is the mean value theorem

Who needs the mean value theorem, anyway?

Theorem

If $f : (a, b) \to \mathbb{R}$ is differentiable and $f' = 0$, then f is constant.

Who needs the mean value theorem, anyway?

Theorem

If $f : (a, b) \to \mathbb{R}$ *is differentiable and* $f' = 0$*, then* f *is constant.*

Proof.

Who needs the mean value theorem, anyway?

Theorem

If $f : (a, b) \to \mathbb{R}$ is differentiable and $f' = 0$, then f is constant.

Proof.

If $f(c) \neq f(d)$ for some $c, d \in (a, b)$,

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If $f(c) \neq f(d)$ for some $c, d \in (a, b)$, then

$$0 \neq \frac{f(c) - f(d)}{c - d}$$

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If $f : (a, b) \to \mathbb{R}$ is differentiable and $f' = 0$, then f is constant.

Proof.

If $f(c) \neq f(d)$ for some $c, d \in (a, b)$, then

$$0 \neq \frac{f(c) - f(d)}{c - d} = f'(\xi) = 0.$$

□

Who needs the mean value theorem, anyway?

Theorem

If $f\colon (a,b) \to \mathbb{R}$ is differentiable and $f' = 0$, then f is constant.

Theorem (L'Hospital's rule, 1696, J. Bernoulli)

If $f, g\colon (a,b) \to \mathbb{R}$ are differentiable, $g' \neq 0$, $\lim_{a+} f = \lim_{a+} g = 0$ and $\lim_{a+} f'/g'$ exists, then

$$\lim_{a+} f/g = \lim_{a+} f'/g'.$$

Who needs the mean value theorem, anyway?

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Proof.

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(\xi)}{g'(\xi)}$$

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$$\lim_{a+} f/g = \lim_{a+} f'/g'.$$

Proof.

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(\xi)}{g'(\xi)}, \qquad x \to a+ \Longrightarrow \xi \to a+$$

□

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$$\lim_{a+} f/g = \lim_{a+} f'/g'.$$

Remark.

It is not difficult to avoid the mean value theorems (Bers, Cohen, Boas).

Historical development of Rolle's theorem

Prehistory:

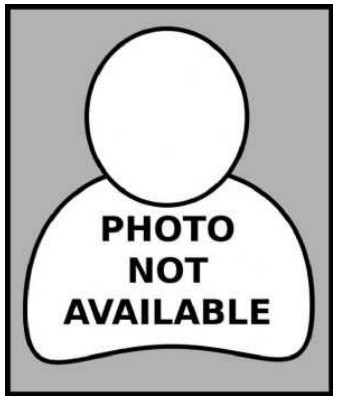
Historical development of Rolle's theorem

Prehistory:

- Bashkara (1114–1185), Parameshvara (1380–1460)
  - astronomical calculations (inverse sine)
  - traces of Rolle's and mean value theorem(?)

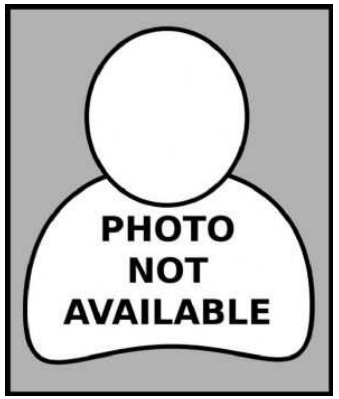
Historical development of Rolle's theorem

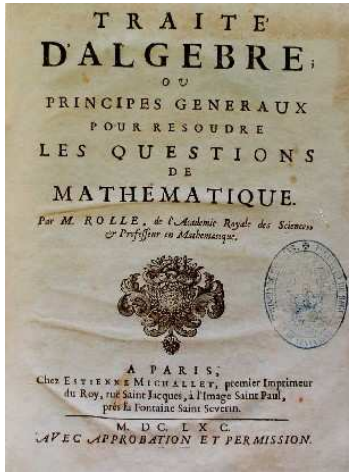
- Michel Rolle (1652–1719)



Historical development of Rolle's theorem

- Michel Rolle (1652–1719)
  - Traité d'algèbre (1690)
    Démonstration d'une méthode pour résoudre les égalitez de tous les degrez (1691) (theorem found in 1910)
  - $f'(x) = 0$ has at least one root between two consecutive roots of $f(x) = 0$ (method of cascades for polynomials)




TRAITE
D'ALGEBRE,
OU
PRINCIPES GENERAUX
POUR RESOUDRE
LES QUESTIONS
DE
MATHEMATIQUE.
Par M. ROLLE, de l'Academie Royale des Sciences & Professeur en Mathematique.
A PARIS,
Chez ESTIENNE MICHALLET, premier Imprimeur du Roy, rue Saint Jacques, à l'Image Saint Paul, près la Fontaine Saint Severin.
M. DC. LXC.
AVEC APPROBATION ET PERMISSION.


DEMONSTRATION
D'UNE
METHODE,
POUR RESOUDRE
LES EGALITEZ
DE TOUS LES DEGREZ,
M. DC. XCI.
AVEC PERMISSION.

Historical development of Rolle's theorem

- Michel Rolle (1691)
- Colin Maclaurin (1698–1746)

Historical development of Rolle's theorem

- Michel Rolle (1691)
- Colin Maclaurin (1698–1746)
  - A second letter...concerning the roots of equations (1729)

THEOREM III. *In general, the Roots of the Equation* $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3}$ &c. $= 0$, *are the* Limits *of the Roots of the Equation* $nx^{n-1} - \overline{n-1} \times Ax^{n-2} + \overline{n-2} \times Bx^{n-3}$ &c. $= 0$, *or of any Equation that is deduced from it by multiplying its Terms by any Arithmetical Progression* $l \mp d, l \mp 2d, l \mp 3d$ &c. *and conversely the Roots of this new Equation will be the* Limits *of the Roots of the proposed Equation* $x^n - Ax^{n-1} + Bx^{n-2}$ &c. $= 0$.

Historical development of Rolle's theorem

- Michel Rolle (1691)
- Colin Maclaurin (1729)
- Leonhard Euler (1707–1783)

Historical development of Rolle's theorem

- Michel Rolle (1691)
- Colin Maclaurin (1729)
- Leonhard Euler (1707–1783)
  - Institutiones calculi differentialis (1755)

246. Si igitur habeatur aequatio algebraica quotcumque dimensionum:

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + \&c. = 0$$

quae duas habeat radices inter se aequales, erit quoque

$$nx^{n-1} + (n-1)Ax^{n-2} + (n-2)Bx^{n-3} + (n-3)Cx^{n-4} + (n-4)Dx^{n-5} + \&c. = 0.$$

Scilicet illius aequationis radix duplex simul erit radix istius aequationis. Multiplicetur illa per n, ab eaque haec per x

Historical development of Rolle's theorem

- Michel Rolle (1691)
- Colin Maclaurin (1729)
- Leonhard Euler (1755)
- Joseph-Louis Lagrange (1736-1813)

Historical development of Rolle's theorem

- Michel Rolle (1691)
- Colin Maclaurin (1729)
- Leonhard Euler (1755)
- Joseph-Louis Lagrange (1736-1813)
  - Traité de la résolution des équations numériques de tous les degrés (1798)

Il est d'abord évident que l'équation $F(x) = 0$ du degré m aura m racines réelles et que l'équation dérivée $F'(x) = 0$ du degré $m - 1$ aura aussi nécessairement $m - 1$ racines réelles, puisque, entre deux racines réelles consécutives de l'équation $F(x) = 0$, il tombe toujours une racine réelle de l'équation $F'(x) = 0$. Par la même raison, la seconde équation dérivée $F''(x) = 0$ aura aussi nécessairement toutes ses racines réelles, et ainsi de suite.

Historical development of Rolle's theorem

- Michel Rolle (1691)
- Colin Maclaurin (1729)
- Leonhard Euler (1755)
- Joseph-Louis Lagrange (1798)
- Moritz Wilhelm Drobisch (1802-1896)

Historical development of Rolle's theorem

- Michel Rolle (1691)
- Colin Maclaurin (1729)
- Leonhard Euler (1755)
- Joseph-Louis Lagrange (1798)
- Moritz Wilhelm Drobisch (1802-1896)
  - Grundzüge der Lehre von den höheren numerischen Gleichungen (1843)

115. 116. Rolle's Sätze von der Begrenzung der reellen Wurzeln der derivirten Gleichung durch die reellen der ursprünglichen, so wie dieser durch jene.

Proof details

Historical development of Rolle's theorem

- Michel Rolle (1691)
- Colin Maclaurin (1729)
- Leonhard Euler (1755)
- Joseph-Louis Lagrange (1798)
- Moritz Wilhelm Drobisch (1802-1896)
  - Grundzüge der Lehre von den höheren numerischen Gleichungen (1843)

nen und daher allgemein den Satz aufstellen: *je zwei nächste reelle Wurzeln einer jeden derivirten Gleichung schliessen eine reelle Wurzel der nächstfolgenden derivirten Gleichung ein und können daher zu Grenzen derselben dienen.*

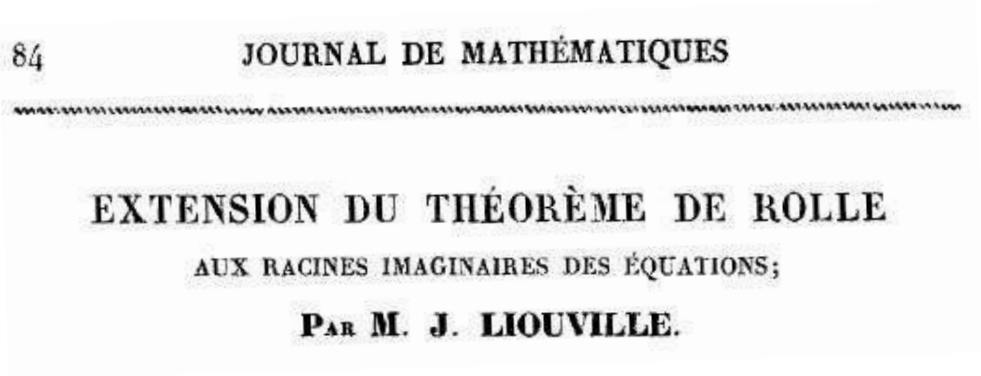
Proof details

Historical development of Rolle's theorem

- Michel Rolle (1691)
- Colin Maclaurin (1729)
- Leonhard Euler (1755)
- Joseph-Louis Lagrange (1798)
- Moritz Wilhelm Drobisch (1843)
- Joseph Liouville (1809–1882)

Historical development of Rolle's theorem

- Michel Rolle (1691)
- Colin Maclaurin (1729)
- Leonhard Euler (1755)
- Joseph-Louis Lagrange (1798)
- Moritz Wilhelm Drobisch (1843)
- Joseph Liouville (1809–1882)
  - Extension du théorème de Rolle aux racines imaginaires des équations (1864)



84 JOURNAL DE MATHÉMATIQUES

EXTENSION DU THÉORÈME DE ROLLE

AUX RACINES IMAGINAIRES DES ÉQUATIONS;

PAR M. J. LIOUVILLE.

Historical development of Rolle's theorem

- Michel Rolle (1691)
- Colin Maclaurin (1729)
- Leonhard Euler (1755)
- Joseph-Louis Lagrange (1798)
- Moritz Wilhelm Drobisch (1843)
- Joseph Liouville (1864)
- Joseph-Alfred Serret (1819–1885)

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- Michel Rolle (1691)
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- Leonhard Euler (1755)
- Joseph-Louis Lagrange (1798)
- Moritz Wilhelm Drobisch (1843)
- Joseph Liouville (1864)
- Joseph-Alfred Serret (1819–1885)
  - Cours d'algèbre supérieure (1866)
  - every continuous function is differentiable

Historical development of Rolle's theorem

- Michel Rolle (1691)
- Colin Maclaurin (1729)
- Leonhard Euler (1755)
- Joseph-Louis Lagrange (1798)
- Moritz Wilhelm Drobisch (1843)
- Joseph Liouville (1864)
- Joseph-Alfred Serret (1819–1885)
  - Cours d'algèbre supérieure (1866)

Théorème de Rolle.

123. La proposition connue sous le nom de *théorème de Rolle* est utile dans quelques circonstances, et elle se rattache directement à la théorie que nous exposons. Aussi croyons-nous devoir la présenter ici.

THÉORÈME. — *Si a et b désignent deux racines consécutives de l'équation $f(x) = 0$, en sorte que cette équation n'ait aucune autre racine comprise entre a et b, l'équation $f'(x) = 0$, obtenue en égalant à zéro la dérivée de $f(x)$, a au moins une racine comprise entre a et b, et quand elle en a plusieurs, le nombre de ces racines est impair.*

Proof details

Historical development of Rolle's theorem

- Michel Rolle (1691)
- Colin Maclaurin (1729)
- Leonhard Euler (1755)
- Joseph-Louis Lagrange (1798)
- Moritz Wilhelm Drobisch (1843)
- Joseph Liouville (1864)
- Joseph-Alfred Serret (1866)

and Rolle's theorem became well-known...

Historical development of the mean value theorem

Prehistory:

Historical development of the mean value theorem

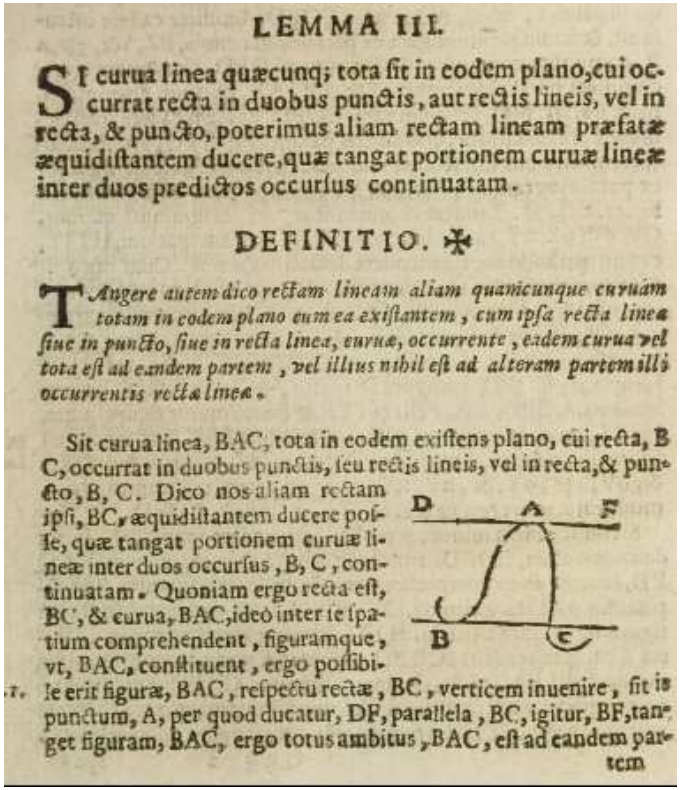
Prehistory:

- Bonaventura Cavalieri (1598-1647)

Historical development of the mean value theorem

Prehistory:

- Bonaventura Cavalieri (1598-1647)
  - Geometria indivisibilibus (1635)



LEMMA III.

SI curua linea quæcunq; tota sit in eodem plano, cui occurrat recta in duobus punctis, aut rectis lineis, vel in recta, & puncto, poterimus aliam rectam lineam præfatæ æquidistantem ducere, quæ tangat portionem curuæ lineæ inter duos predictos occursus continuatam.

DEFINITIO. ✠

TAngere autem dico rectam lineam aliam quamcunque curuam totam in eodem plano cum ea existantem, cum ipsa recta linea sive in puncto, sive in recta linea, curuæ, occurrente, eadem curua vel tota est ad eandem partem, vel illius nihil est ad alteram partem illis occurrentis rectæ lineæ.

Sit curua linea, BAC, tota in eodem existens plano, cui recta, BC, occurrat in duobus punctis, seu rectis lineis, vel in recta, & puncto, B, C. Dico nos aliam rectam ipsi, BC, æquidistantem ducere posse, quæ tangat portionem curuæ lineæ inter duos occursus, B, C, continuatam. Quoniam ergo recta est, BC, & curua, BAC, ideò inter se spatium comprehendent, figuramque, vt, BAC, constituent, ergo possibile erit figuræ, BAC, respectu rectæ, BC, verticem inuenire, sit is punctum, A, per quod ducatur, DF, parallela, BC, igitur, BF, tanget figuram, BAC, ergo totus ambitus, BAC, est ad eandem partem

Historical development of the mean value theorem

Prehistory:

- Bonaventura Cavalieri
  - Geometria indivisibilibus (1635)

[2] *Ibid.*: Si curva linea quaecunque tota sit in eodem plano, cui occurrat recta in duobus punctis, aut rectis lineis, vel in recta, & puncto, poterimus aliam rectam lineam praefatae aequidistantem ducere, quae tangat portionem curvae lineae inter duos predictos occursus continuatam.

If a curved line is situated in one plane and if a straight line meets it in either two points, two line segments, or in a line segment and a point, then we can draw another straight line parallel to the previous line which touches the part of the curve situated between the two mentioned meetings.[2]

Historical development of the mean value theorem

- Joseph-Louis Lagrange (1736-1813)

Historical development of the mean value theorem

- Joseph-Louis Lagrange (1736-1813)
  - Leçons sur le calcul des fonctions (1801)

Soit D une quantité donnée qu'on pourra prendre aussi petite qu'on voudra; on pourra donc toujours donner à i une valeur assez petite pour que la valeur de V soit renfermée entre les limites D et $-$D: donc, puisqu'on a

$$f(x+i)-f(x)=i[f'(x)+\mathrm{V}],$$

il s'ensuit que la quantité $f(x+i)-f(x)$ sera renfermée entre ces deux-ci

$$i[f'(x)\pm\mathrm{D}].$$

Historical development of the mean value theorem

- Joseph-Louis Lagrange (1801)
- André-Marie Ampère (1775–1836)

Historical development of the mean value theorem

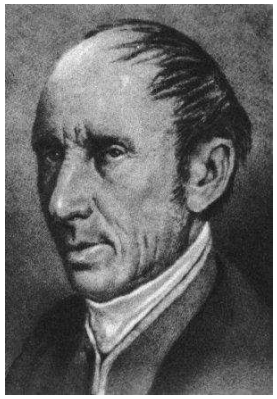
- Joseph-Louis Lagrange (1801)
- André-Marie Ampère (1775–1836)
  - Recherches sur quelques point de la théorie des fonctions dérivées (1806)
  - every function is differentiable(?)

est égal à $f'(x)$, plus une fonction I de x et de i qui s'évanouit quand $i = 0$; on a donc

$$f(x + i) = fx + if'(x) + iI.$$

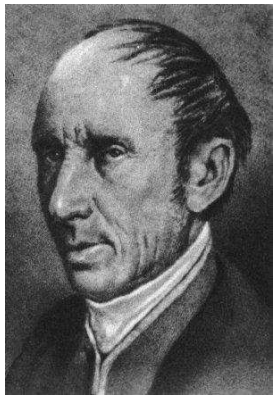
Historical development of the mean value theorem

- Joseph-Louis Lagrange (1801)
- André-Marie Ampère (1806)
- Augustin-Louis Cauchy (1789–1857)



Historical development of the mean value theorem

- Joseph-Louis Lagrange (1801)
- André-Marie Ampère (1806)
- Augustin-Louis Cauchy (1789–1857)
  - Résumé des leçons sur le calcul infinitésimal (1823)



THÉORÈME. — *Si, la fonction $f(x)$ étant continue entre les limites $x = x_0$, $x = \mathrm{X}$, on désigne par* A *la plus petite, et par* B *la plus grande des valeurs que la fonction dérivée $f'(x)$ reçoit dans cet intervalle, le rapport aux différences finies*

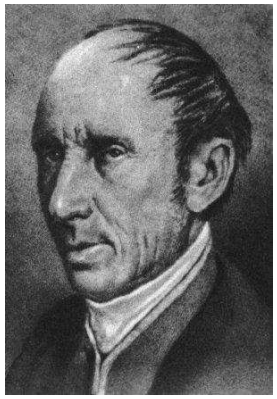
$$\frac{f(\mathrm{X}) - f(x_0)}{\mathrm{X} - x_0} \tag{4}$$

sera nécessairement compris entre A *et* B.

Proof details

Historical development of the mean value theorem

- Joseph-Louis Lagrange (1801)
- André-Marie Ampère (1806)
- Augustin-Louis Cauchy (1789–1857)
  - Résumé des leçons sur le calcul infinitésimal (1823)



Or il est aisé de voir que des raisonnements entièrement semblables à ceux dont nous avons fait usage pour démontrer l'équation précédente suffiront pour établir la formule

$$(1) \qquad \frac{f(X)-f(x_0)}{F(X)-F(x_0)} = \frac{f'[x_0+\theta(X-x_0)]}{F'[x_0+\theta(X-x_0)]},$$

θ désignant encore un nombre inférieur à l'unité, et $F(x)$ une fonction nouvelle qui, toujours croissante ou décroissante depuis la limite $x = x_0$ jusqu'à la limite $x = X$, reste continue, avec sa dérivée $F'(x)$, entre ces mêmes limites.

Proof details

Historical development of the mean value theorem

- Joseph-Louis Lagrange (1801)
- André-Marie Ampère (1806)
- Augustin-Louis Cauchy (1823)
- Pierre-Ossian Bonnet (1819–1892)

Historical development of the mean value theorem

- Joseph-Louis Lagrange (1801)
- André-Marie Ampère (1806)
- Augustin-Louis Cauchy (1823)
- Pierre-Ossian Bonnet (1819–1892)
  - Serret, Cours de calcul infinitésimal (1868)

Théorème I. — *Soit $f(x)$ une fonction de x qui reste continue pour les valeurs de x comprises entre des limites données, et qui, pour ces valeurs, ait une dérivée $f'(x)$ déterminée. Si x_0 et X désignent deux valeurs de x comprises entre les mêmes limites, on aura*

$$\frac{f(\mathrm{X}) - f(x_0)}{\mathrm{X} - x_0} = f'(x_1),$$

x_1 étant une valeur comprise entre x_0 et X.

Proof details

Historical development of the mean value theorem

- Joseph-Louis Lagrange (1801)
- André-Marie Ampère (1806)
- Augustin-Louis Cauchy (1823)
- Pierre-Ossian Bonnet (1819–1892)

  - Serret, Cours de calcul infinitésimal (1868)

THÉORÈME IV. — *Soient $f(x)$ et $F(x)$ deux fonctions de x qui restent continues pour les valeurs de x comprises entre des limites données, et qui, pour ces valeurs, aient des dérivées déterminées $f'(x)$, $F'(x)$. Si x_0 et X désignent deux valeurs de x comprises entre les mêmes limites, et que la dérivée $F'(x)$, qui peut être nulle ou infinie pour $x = x_0$ ou pour $x = X$, ne le soit pas pour les valeurs intermédiaires, on aura*

$$\frac{f(X) - f(x_0)}{F(X) - F(x_0)} = \frac{f'(x_1)}{F'(x_1)},$$

x_1 étant une valeur comprise entre x_0 et X.

Proof details

Historical development of the mean value theorem

- Joseph-Louis Lagrange (1801)
- André-Marie Ampère (1806)
- Augustin-Louis Cauchy (1823)
- Pierre-Ossian Bonnet (1868)
- Ulisse Dini (1845–1918)

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- André-Marie Ampère (1806)
- Augustin-Louis Cauchy (1823)
- Pierre-Ossian Bonnet (1868)
- Ulisse Dini (1845–1918)
  - Fondamenti per la teorica delle funzioni de variabli reali (1878)
  - perfect proof (and perfect book)

Historical development of the mean value theorem

- Joseph-Louis Lagrange (1801)
- André-Marie Ampère (1806)
- Augustin-Louis Cauchy (1823)
- Pierre-Ossian Bonnet (1868)
- Ulisse Dini (1878)
- Carl Gustav Axel Harnack (1851–1888)

Historical development of the mean value theorem

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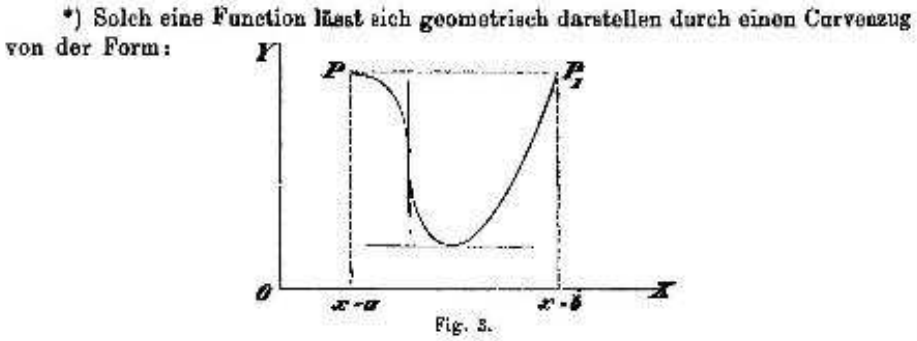
- André-Marie Ampère (1806)
- Augustin-Louis Cauchy (1823)
- Pierre-Ossian Bonnet (1868)
- Ulisse Dini (1878)
- Carl Gustav Axel Harnack (1851–1888)
  - Die Elemente der Differential- und Integralrechnung (1881)



*) Solch eine Function läßt sich geometrisch darstellen durch einen Curvenzug von der Form:

Fig. 3.

Historical development of the mean value theorem

- Joseph-Louis Lagrange (1801)
- André-Marie Ampère (1806)
- Augustin-Louis Cauchy (1823)
- Pierre-Ossian Bonnet (1868)
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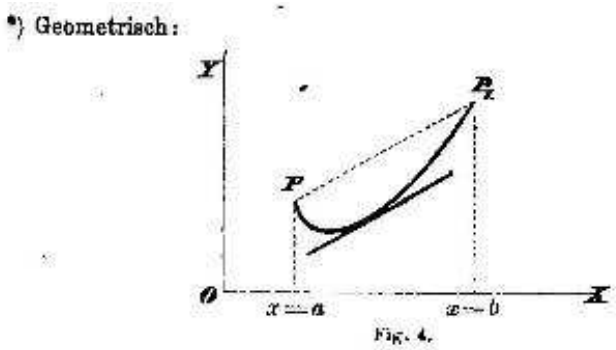

Historical development of the mean value theorem

- Joseph-Louis Lagrange (1801)
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- Augustin-Louis Cauchy (1823)
- Pierre-Ossian Bonnet (1868)
- Ulisse Dini (1878)
- Carl Gustav Axel Harnack (1881)
- Moritz Pasch (1843–1930)

Historical development of the mean value theorem

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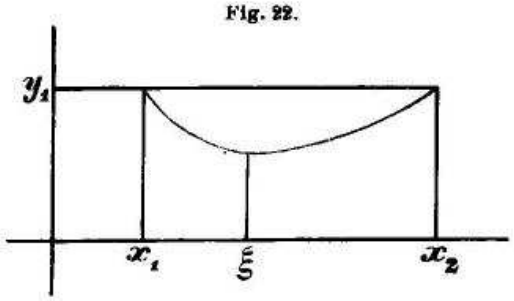
- Augustin-Louis Cauchy (1823)
- Pierre-Ossian Bonnet (1868)
- Ulisse Dini (1878)
- Carl Gustav Axel Harnack (1881)
- Moritz Pasch (1843–1930)
  - Einleitung in die Differential- und Integralrechnung (1882)



Historical development of the mean value theorem

- Joseph-Louis Lagrange (1801)
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- Augustin-Louis Cauchy (1823)
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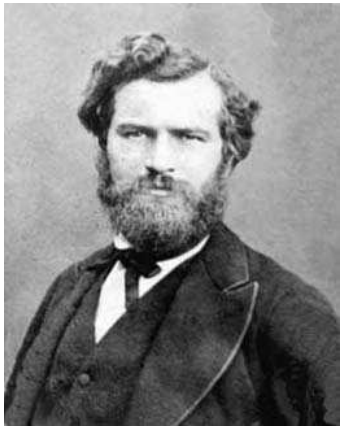
- Ulisse Dini (1878)
- Carl Gustav Axel Harnack (1881)
- Moritz Pasch (1882)

and the theorem seemed to be perfectly clear...

Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)



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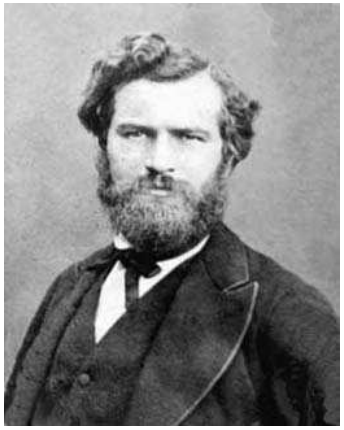
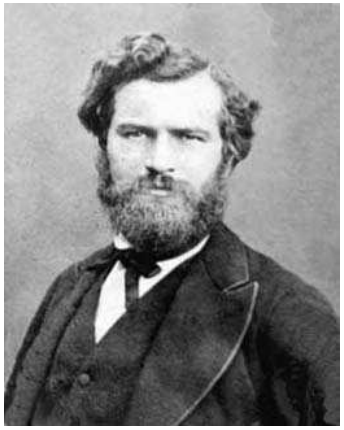


Cours d'analyse (1882) contains an erroneous proof

Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)

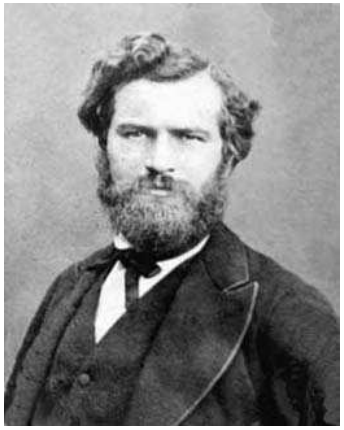


Cours d'analyse (1882) contains an erroneous proof

Proof of the mean value inequality.

Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)



Cours d'analyse (1882) contains an erroneous proof

Proof of the mean value inequality.

Take $a = a_0 < a_2 < \cdots < a_n = b$.

Correspondence of Peano and Jordan

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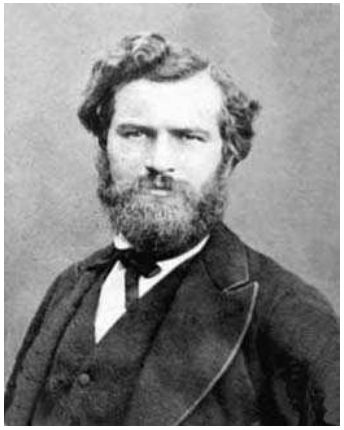


Cours d'analyse (1882) contains an erroneous proof

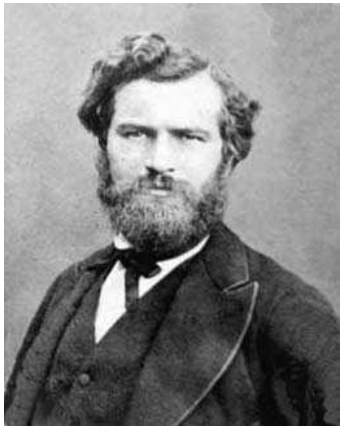
Proof of the mean value inequality.

Take $a = a_0 < a_2 < \cdots < a_n = b$. Then

$$f(a_i) - f(a_{i-1}) = (a_i - a_{i-1})[f'(a_{i-1}) + \varepsilon_i].$$

Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)



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Take $a = a_0 < a_2 < \cdots < a_n = b$. Then

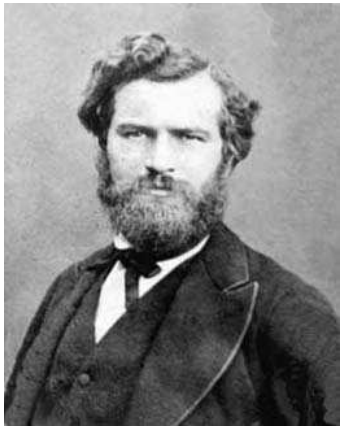
$$f(a_i) - f(a_{i-1}) = (a_i - a_{i-1})[f'(a_{i-1}) + \varepsilon_i].$$

Thus, with $M = \max_{[a,b]} |f'|$, $m = \min_{[a,b]} |f'|$, $\varepsilon = \max_{i=1,\dots,n} |\varepsilon_i|$, after summation

$$(m - \varepsilon)(b - a) \leq f(b) - f(a) \leq (M + \varepsilon)(b - a).$$

Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)



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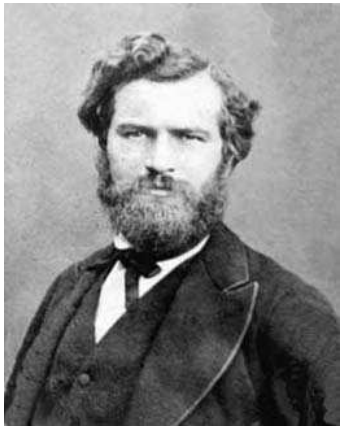
Thus, with $M = \max_{[a,b]} |f'|$, $m = \min_{[a,b]} |f'|$, $\varepsilon = \max_{i=1,\dots,n} |\varepsilon_i|$, after summation

$$(m - \varepsilon)(b - a) \leq f(b) - f(a) \leq (M + \varepsilon)(b - a).$$

Let $n \to \infty$, then $\varepsilon \to 0$. □

Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)



- Giuseppe Peano (1858–1932)

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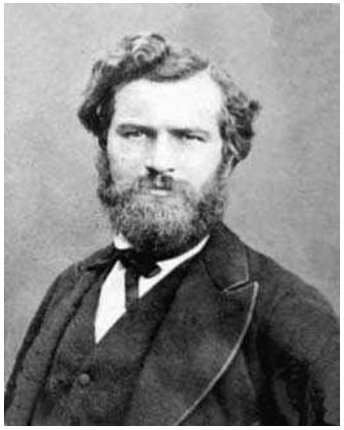

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Letter to Jordan (Nouv. Ann. Math., 1884)

Correspondence of Peano and Jordan

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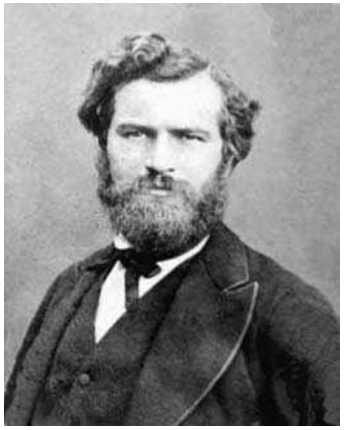


- Giuseppe Peano (1858–1932)

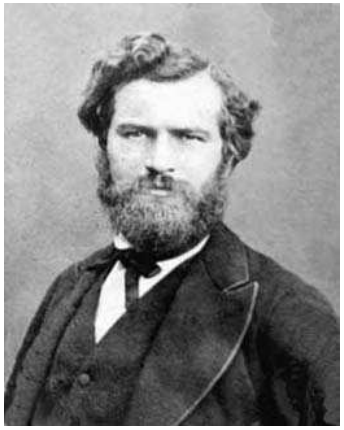
Letter to Jordan (Nouv. Ann. Math., 1884)

Problem.

If $f(x) = x^2 \sin \frac{1}{x}$, $a_0 = 0$, $a_1 = 1/2n\pi$, $a_2 = 1/(2n+1)\pi$ then $f(a_1) = f(a_2) = 0$, $f'(a_1) = -1$ and so $\varepsilon_2 = 1$.

Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)



- Giuseppe Peano (1858–1932)

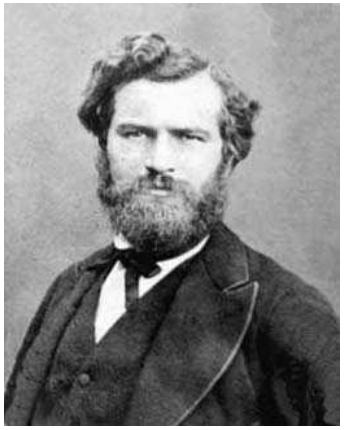
Letter to Jordan (Nouv. Ann. Math., 1884)

Problem.

The continuity of the derivative is used implicitly in your proof. (Exactly the same error made in Jules Hoüel's Cours de calcul infinitésimal (1879).)

Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)



- Giuseppe Peano (1858–1932)

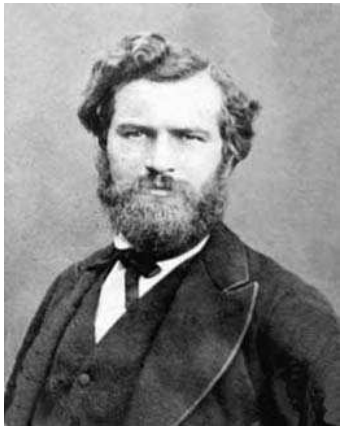
Letter to Jordan (Nouv. Ann. Math., 1884)

Problem.

However, the theorem can be easily proved without assuming the continuity of the derivative.

Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)

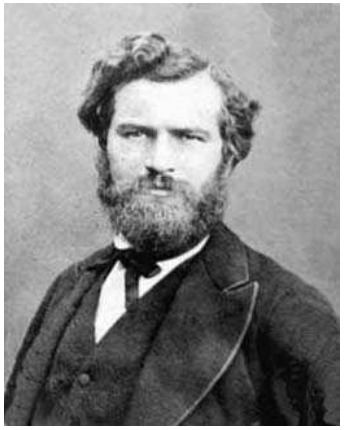


Reply to Peano (1884).

- Giuseppe Peano (1858–1932)

Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)



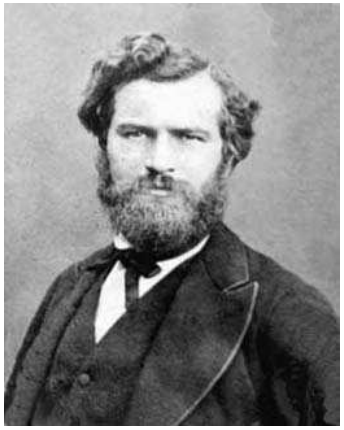
Reply to Peano (1884).

I know that my proof requires the continuity of the derivative. I will correct the error in Volume 3 of my book.

- Giuseppe Peano (1858–1932)

Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)



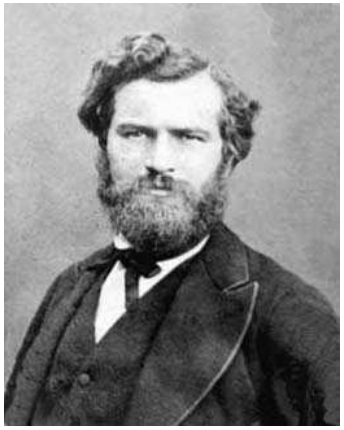
Reply to Peano (1884).

However, I would be very happy if you send me a proof without that assumption, because I do not know any.

- Giuseppe Peano (1858–1932)

Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)

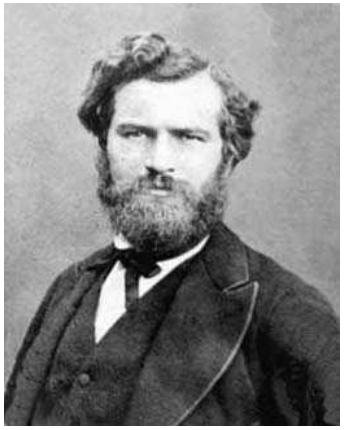


- Giuseppe Peano (1858–1932)

Reply to Jordan (1884).

Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)



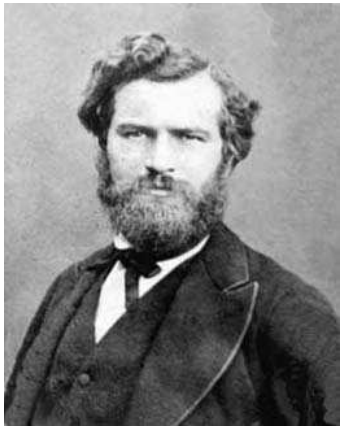
- Giuseppe Peano (1858–1932)

Reply to Jordan (1884).

I learned it from my teacher Angelo Genocchi. It is due to O. Bonnet, see Serret (with imperfections) and Dini, Harnack, Pasch (perfectly).

Correspondence of Peano and Jordan

- Camille Jordan (1838–1922)



Jordan's book is modified accordingly (1887).

34. *Corollaire.* — Soient $f(x)$, $\varphi(x)$, $\psi(x)$ trois fonctions quelconques admettant des dérivées dans l'intervalle de a à $a+h$; considérons le déterminant

$$\begin{vmatrix} f(a) & \varphi(a) & \psi(a) \\ f(a+h) & \varphi(a+h) & \psi(a+h) \\ f(a+\theta h) & \varphi(a+\theta h) & \psi(a+\theta h) \end{vmatrix}.$$

C'est une fonction de θ qui s'annule pour $\theta = 0$ et $\theta = 1$, et qui admet, pour dérivée dans cet intervalle, le produit de h par le déterminant

$$\Delta = \begin{vmatrix} f(a) & \varphi(a) & \psi(a) \\ f(a+h) & \varphi(a+h) & \psi(a+h) \\ f'(a+\theta h) & \varphi'(a+\theta h) & \psi'(a+\theta h) \end{vmatrix}.$$

Ce nouveau déterminant devra donc s'annuler pour une valeur de θ comprise entre 0 et 1.

35. La démonstration que nous avons donnée de cette formule (*Calcul différentiel*, n° 15) supposait inutilement la continuité de la dérivée $f'(x)$; elle repose, en outre, sur un autre postulatum que nous n'avons pas formulé explicitement, à savoir que $\frac{f(x+\Delta x)-f(x)}{\Delta x}$ tend uniformément vers $f'(x)$ quand Δx tend vers zéro pour toutes les valeurs de x comprises entre a et $a+h$. En effet, s'il en était autrement, on ne pourrait affirmer, comme nous l'avons fait, que

$$\frac{f(a_{k+1})-f(a_k)}{a_{k+1}-a_k} - f'(a_k) = \varepsilon_k$$

tend vers zéro avec $a_{k+1}-a_k$. Il en serait effectivement ainsi, d'après la définition même de la dérivée, si $a_{k+1}-a_k$ variait seul, a_k ayant une valeur fixe quelconque; mais ici a_k varie nécessairement en même temps que $a_{k+1}-a_k$, à mesure qu'on fait croître le nombre des intervalles partiels $aa_1, \ldots, a_k a_{k+1}, \ldots$.

- Giuseppe Peano (1858–1932)

Correspondence of Peano and Gilbert

- Louis-Philippe Gilbert (1832–1892)

Chair of analysis at Université Catholique de Louvain
(and cousin of Charles de la Vallée Poussin)

- Giuseppe Peano (1858–1932)

Correspondence of Peano and Gilbert

- Louis-Philippe Gilbert (1832–1892)

Reply to Peano (1884).

- Giuseppe Peano (1858–1932)

Correspondence of Peano and Gilbert

- Louis-Philippe Gilbert (1832–1892)

Reply to Peano (1884).

Without continuity of the derivative, the theorem is false.

- Giuseppe Peano (1858–1932)

Correspondence of Peano and Gilbert

- Louis-Philippe Gilbert (1832–1892)

Reply to Peano (1884).

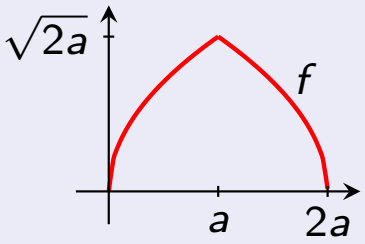
If $f(x) = \sqrt{2x}$ $(x \in (0, a))$ and $f(x) = \sqrt{2(2a - x)}$ $(x \in (a, 2a))$, then $f(a + h) - f(a - h) = 0$ but there is no ξ for which $f'(\xi) = 0$.

- Giuseppe Peano (1858–1932)

Correspondence of Peano and Gilbert

- Louis-Philippe Gilbert (1832–1892)

Reply to Peano (1884).



- Giuseppe Peano (1858–1932)

Correspondence of Peano and Gilbert

- Louis-Philippe Gilbert (1832–1892)

- Giuseppe Peano (1858–1932)

Reply to Gilbert (1884).

Correspondence of Peano and Gilbert

- Louis-Philippe Gilbert (1832–1892)

- Giuseppe Peano (1858–1932)

Reply to Gilbert (1884).

Your function is not differentiable at some point.
Differentiability means equal left and right derivative.

Correspondence of Peano and Gilbert

- Louis-Philippe Gilbert (1832–1892)

- Giuseppe Peano (1858–1932)

Reply to Gilbert (1884).

The proof is easy, I learned it from my teacher A. Genocchi. It is due to Bonnet, see Serret (with imperfections) and Dini, Harnack, Pasch (perfectly).

Correspondence of Peano and Gilbert

- Louis-Philippe Gilbert (1832–1892)

- Giuseppe Peano (1858–1932)

Reply to Gilbert (1884).

And, without malice, I give you an exercise: show that if f is differentiable then for all $\varepsilon > 0$ there is a partition $a = a_0 < \cdots < a_n = b$ such that $\left|\frac{f(a_{i+1})-f(a_i)}{a_{i+1}-a_i} - f'(a_i)\right| < \varepsilon$.

Correspondence of Peano and Gilbert

- Louis-Philippe Gilbert (1832–1892)

Reply to Peano (1884).

- Giuseppe Peano (1858–1932)

Correspondence of Peano and Gilbert

- Louis-Philippe Gilbert (1832–1892)

Reply to Peano (1884).

I misunderstood your concept of discontinuity of derivative. Of course, I knew Bonnet's proof.

- Giuseppe Peano (1858–1932)

Correspondence of Peano and Gilbert

- Louis-Philippe Gilbert (1832–1892)

Reply to Peano (1884).

Your exercise is not interesting for me at all, however, I provide you a solution.

- Giuseppe Peano (1858–1932)

References

- Judith V. Grabiner, *The Origins of Cauchy's Rigor in Calculus*, MIT Press, Cambridge, 1981.
- June Barrow-Green, From cascades to calculus: Rolle's Theorem, In: Robson, Eleanor and Stedall, Jacqueline eds. The Oxford Handbook of the History of Mathematics, Oxford Handbooks in Mathematics, Oxford University Press, Oxford, 737-754., 2009.
- Jean Mawhin, Some contributions of Peano to analysis in the light of the work of Belgian mathematicians, In: Fulvia Skof, Giuseppe Peano between mathematics and logic, Springer Italia, Milan, 13-28., 2009.
- Gallica, Bibliothèque nationale de France
- Internet Archive (Digital Library)

The End

Thank you for your attention!

Drobisch's proof of Rolle's theorem

Proof.

Return

Drobisch's proof of Rolle's theorem

Proof.

Let $f(x) = (x - \alpha_1) \dots (x - \alpha_n)\varphi(x)$ where $\alpha_1 < \dots < \alpha_n$ and $\varphi > 0$.

Return

Drobisch's proof of Rolle's theorem

Proof.

Let $f(x) = (x - \alpha_1)\dots(x - \alpha_n)\varphi(x)$ where $\alpha_1 < \dots < \alpha_n$ and $\varphi > 0$. Then

$$\begin{aligned} f'(\alpha_1) &= (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)\dots(\alpha_1 - \alpha_n)\varphi(\alpha_1), \\ f'(\alpha_2) &= (\alpha_2 - \alpha_1)(\alpha_2 - \alpha_3)\dots(\alpha_2 - \alpha_n)\varphi(\alpha_2), \\ f'(\alpha_3) &= (\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2)\dots(\alpha_3 - \alpha_n)\varphi(\alpha_3), \\ &\vdots \end{aligned}$$

Return

Drobisch's proof of Rolle's theorem

Proof.

Let $f(x) = (x - \alpha_1)\dots(x - \alpha_n)\varphi(x)$ where $\alpha_1 < \dots < \alpha_n$ and $\varphi > 0$. Then

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Therefore, $f'(\alpha_i)$ and $f'(\alpha_{i+1})$ have different signs

Return

Drobisch's proof of Rolle's theorem

Proof.

Let $f(x) = (x - \alpha_1)\dots(x - \alpha_n)\varphi(x)$ where $\alpha_1 < \dots < \alpha_n$ and $\varphi > 0$. Then

$$\begin{aligned} f'(\alpha_1) &= (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)\dots(\alpha_1 - \alpha_n)\varphi(\alpha_1), \\ f'(\alpha_2) &= (\alpha_2 - \alpha_1)(\alpha_2 - \alpha_3)\dots(\alpha_2 - \alpha_n)\varphi(\alpha_2), \\ f'(\alpha_3) &= (\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2)\dots(\alpha_3 - \alpha_n)\varphi(\alpha_3), \\ &\vdots \end{aligned}$$

Therefore, $f'(\alpha_i)$ and $f'(\alpha_{i+1})$ have different signs thus f' has a root between α_i and α_{i+1}. □

Return

Serret's proof of Rolle's theorem

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thus $\frac{f(a+h)}{f'(a+h)} > 0$ for small h and so $f'(a+h) > 0$. Similarly, $f'(b-h) < 0$ for small h. Therefore, f' has a root between $a+h$ and $b-h$. □

Return

Cauchy's proof of the mean value inequality

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Let $\varepsilon > 0$ and $\delta > 0$ be such that if $0 < i < \delta$ (i=indeterminate), then

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$$\min_{[a,b]} f' - \varepsilon \leq f'(x_i) - \varepsilon < \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} < f'(x_i) + \varepsilon \leq \max_{[a,b]} f' + \varepsilon,$$

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and so by averaging with weights $(x_i - x_{i-1})$ one obtains

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and so by averaging with weights $(x_i - x_{i-1})$ one obtains

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The assertion follows by $\varepsilon \to 0$. □

Return

Cauchy's proof of the mean value inequality

Remark.

- Problem: δ depends on x
  (one needs the continuity of the derivative, as later noted Peano).
- The notion of mean (moyenne) was introduced by Cauchy in Analyse algébrique (1821).
- Cauchy proved rigorously the intermediate value theorem in Analyse algébrique (1821) so the mean value thereom also follows.

Return

Cauchy's proof of the generalized mean value theorem

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Assume $g' > 0$ and let $A := \min_{[a,b]} f'/g'$, $B := \max_{[a,b]} f'/g'$.

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$$f'(x) - Ag'(x) \geq 0 \quad \text{and} \quad Bg'(x) - f'(x) \geq 0 \quad (x \in (a,b)).$$

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Therefore $(f - Ag)$ and $(Bg - f)$ are increasing thus

$$f(b) - f(a) - A(g(b) - g(a)) \geq 0 \quad \text{and} \quad B(g(b) - g(a)) - (f(b) - f(a)) \geq 0.$$

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Therefore $(f - Ag)$ and $(Bg - f)$ are increasing thus

$$f(b)-f(a)-A(g(b)-g(a)) \geq 0 \quad \text{and} \quad B(g(b)-g(a))-(f(b)-f(a)) \geq 0.$$

So

$$A(g(b) - g(a)) \leq f(b) - f(a) \leq B(g(b) - g(a)),$$

and

$$A \leq \frac{f(b) - f(a)}{g(b) - g(a)} \leq B$$

which is Cauchy's mean value inequality.

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Therefore $(f - Ag)$ and $(Bg - f)$ are increasing thus

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and

$$A \leq \frac{f(b) - f(a)}{g(b) - g(a)} \leq B$$

which is Cauchy's mean value inequality. Cauchy's mean value theorem follows by the intermediate value theorem and the continuity of f'/g'. □

Return

Bonnet's proof of the mean value theorem

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Denote $A = \frac{f(b)-f(a)}{b-a}$ and $\varphi(x) = [f(x) - Ax] - [f(a) - Aa]$.

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Denote $A = \frac{f(b)-f(a)}{b-a}$ and $\varphi(x) = [f(x) - Ax] - [f(a) - Aa]$. Then $\varphi(a) = \varphi(b) = 0$. Therefore, if φ is not constant, it has an extremum at some $x_1 \in (a, b)$ and there $\varphi'(x_1) = 0$. Otherwise, $\varphi' = 0$. $\square$

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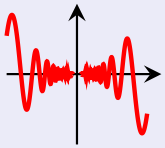
Proof.

Denote $A = \frac{f(b)-f(a)}{b-a}$ and $\varphi(x) = [f(x) - Ax] - [f(a) - Aa]$. Then $\varphi(a) = \varphi(b) = 0$. Therefore, if φ is not constant, it has an extremum at some $x_1 \in (a, b)$ and there $\varphi'(x_1) = 0$. Otherwise, $\varphi' = 0$. □

Remark.

- No reference to Rolle's theorem, but a complete proof.
- minor "imperfection" (as criticized by Peano):
  "if φ is not constant, then it should increase or decrease at a"
  is not true, for instance, $x^2 \sin \frac{1}{x}$ at $x = 0$.



- Analogous proof for Cauchy's mean value theorem: $A = \frac{f(b)-f(a)}{g(b)-g(a)}$ and $\varphi(x) = [f(x) - Ag(x)] - [f(a) - Ag(a)]$.

Return