

Successive iterations and logarithmic means

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Abstract

The successive iteration (started by Lagrange and Gauss) produces a new mean from two given ones. There are several examples of matrix means and this needs the proof of the matrix monotonicity of the corresponding representing function. The paper contains extensions of the logarithmic mean and it is obtained that the Stolarsky mean can be used also for matrices. There are examples of complementary means.

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A continuous function $m: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is called a mean if

$$\min(x, y) \leq m(x, y) \leq \max(x, y) \quad (x, y \in \mathbb{R}^+). \quad (1)$$

A mean is symmetric if $m(x, y) = m(y, x)$ for $x, y \in \mathbb{R}^+$ and it is strict if both inequalities in (1) are strict for $x \neq y$. Means may be obtained also from the recursively defined double sequence which is a successive iteration of two means. In the paper we consider these iterations and some corresponding means, such as the logarithmic and the Stolarsky mean, for matrices. The theory of successive iteration of means began at the end of the 18th century by the works of J.-L. Lagrange and C. F. Gauss.

If we consider the iteration

$$\begin{aligned} a_0 &:= a, & b_0 &:= b, \\ a_{n+1} &:= \frac{a_n + b_n}{2}, & b_{n+1} &:= \sqrt{a_n b_n}, \end{aligned} \quad (2)$$

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then the sequences $(a_n), (b_n)$ converge to a common limit which is the so-called **Gauss's arithmetic-geometric mean** $\mathbf{AG}(a, b)$ with the characterization

$$\frac{1}{\mathbf{AG}(a, b)} = \frac{2}{\pi} \int_0^\infty \frac{dt}{\sqrt{(a^2 + t^2)(b^2 + t^2)}}, \quad (3)$$

see [7, 4]. The recurrence (2) was first considered by Lagrange in 1785 and later by Gauss in 1791 at the age of 14 who discovered the formula (3) in 1799, see [6, 7].

1 Gaussian matrix mean process

The mean of numbers can be extended to matrices if $f(x) = m(1, x)$ is a matrix monotone function, that is, $0 < A \leq B$ implies $f(A) \leq f(B)$ for every positive matrices A, B . If $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a matrix monotone function with the property $f(1) = 1$, then

$$m_f(A, B) = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}$$

is a matrix mean for $A, B > 0$ and f is called the representing function of m_f , see [9]. The following conditions give an axiomatic approach:

- (i) $m_f(A, A) = A$ for every A ,
- (ii) if $A \leq A'$ and $B \leq B'$, then $m_f(A, B) \leq m_f(A', B')$ (joint monotonicity),
- (iii) m is continuous,
- (iv) $C m_f(A, B) C^* \leq m_f(CAC^*, CBC^*)$ (transformer inequality).

Obviously, condition (i) is equivalent to $f(1) = 1$. Furthermore, it is easily seen that the symmetry of the mean is equivalent to the condition $xf(x^{-1}) = f(x)$. From the symmetry property after differentiation it follows that

$$f'(x) = f(x^{-1}) - x^{-1} f'(x^{-1})$$

hence by substituting $x = 1$ one obtains $f'(1) = 1/2$. It was proved in [9] that if f is a representing function of a symmetric matrix mean then

$$\frac{2x}{x+1} \leq f(x) \leq \frac{x+1}{2},$$

that is, a matrix mean is between the arithmetic and harmonic means.

The arithmetic-geometric mean can be generalized even for matrix means. If $f(x) = \mathbf{AG}(1, x)$ then from (3) the matrix monotonicity is not clear, however the successive iteration gives a matrix mean and this fact implies the matrix monotonicity of $f(x)$.

For positive numbers the **Gaussian double-mean process** is the following:

$$\begin{aligned} a_0 &:= a, & b_0 &:= b, \\ a_{n+1} &:= m_1(a_n, b_n), & b_{n+1} &:= m_2(a_n, b_n) \end{aligned} \quad (4)$$

where m_1 and m_2 are means. In [6] the convergence is proved if the means are strict. The characterization of the limit is the following.

Theorem 1.1. (Invariance principle) *Suppose that for every $a, b > 0$ the recurrences (4) converge. Then the limit $\Phi(a, b)$ is the unique mean satisfying the so-called invariance equation*

$$\Phi(m_1(a, b), m_2(a, b)) = \Phi(a, b) \quad (a, b > 0).$$

In the paper [9] the convergence of (4) is proved also for positive matrices, if m_1 and m_2 are matrix means. If m_1 and m_2 are symmetric we have a simple proof.

Theorem 1.2. *Let m_1 and m_2 be symmetric matrix means. For positive matrices A and B set a recursion*

$$A_0 = A, \quad B_0 = B, \quad A_{n+1} = m_1(A_n, B_n), \quad B_{n+1} = m_2(A_n, B_n). \quad (5)$$

Then (A_n) and (B_n) converge to the same operator mean $m(A, B)$.

Proof. From the inequality

$$m_i(X, Y) \leq \frac{X + Y}{2} \quad (6)$$

we have

$$A_{n+1} + B_{n+1} = m_1(A_n, B_n) + m_2(A_n, B_n) \leq A_n + B_n.$$

Therefore, the decreasing positive sequence has a limit:

$$A_n + B_n \rightarrow X \text{ as } n \rightarrow \infty. \quad (7)$$

Moreover,

$$c_{n+1} := \|A_{n+1}\|_2^2 + \|B_{n+1}\|_2^2 \leq \|A_n\|_2^2 + \|B_n\|_2^2 - \frac{1}{2}\|A_n - B_n\|_2^2 \leq c_n.$$

Thus the numerical sequence (c_n) is decreasing, it has a limit and it follows that

$$\|A_n - B_n\|_2^2 \rightarrow 0$$

and $(A_n), (B_n) \rightarrow X/2$ as $n \rightarrow \infty$. □

We can also study the rate of convergence. Suppose that A and B are strictly positive, m_1, m_2 are symmetric means and $f_i(x) = m_i(1, x)$ are twice differentiable. Then $f'_i(1) = 1/2$ so that the Taylor expansion around $x = 1$ yields

$$f_i(Y) = I + \frac{1}{2}(Y - I) + \frac{f''_i(1)}{2}(Y - I)^2 + o(\|Y - I\|^2).$$

Whence

$$\begin{aligned} A_{n+1} - B_{n+1} &= \frac{f''_1(1) - f''_2(1)}{2} A_n^{1/2} (A_n^{-1/2} B_n A_n^{-1/2} - I)^2 A_n^{1/2} \\ &\quad + A_n^{-1/2} o(\|A_n^{-1/2} B_n A_n^{-1/2} - I\|^2) A_n^{-1/2} \end{aligned}$$

and

$$\begin{aligned} \|A_{n+1} - B_{n+1}\| &\leq \frac{|f''_1(1) - f''_2(1)|}{2} \|A_n^{1/2}\|^2 \|A_n^{-1/2} B_n A_n^{-1/2} - I\|^2 \\ &\quad + \|A_n^{-1/2}\|^2 o(\|A_n^{-1/2} B_n A_n^{-1/2} - I\|^2). \end{aligned}$$

Thus

$$\limsup_{n \rightarrow \infty} \frac{\|A_{n+1} - B_{n+1}\|}{\|A_n - B_n\|^2} < \infty$$

which means that the convergence is quadratic.

Example 1.3. If m_1 is the arithmetic mean and m_2 is the geometric mean, then we can present another formulation of the arithmetic-geometric mean with block matrices. Let $A, B > 0$,

$$X_0 := \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

and

$$X_{n+1} := \frac{1}{2} (X_n)_{11}^{1/2} \left(\sqrt[4]{(X_n)_{11}^{-1/2} X_n (X_n)_{11}^{-1/2}} \right)^2 (X_n)_{11}^{1/2},$$

where $\sqrt[4]{X}$ of the block matrix is defined in the following form:

$$\sqrt[4]{X} := \begin{bmatrix} \sqrt{X_{11}} & \sqrt{X_{12}} \\ \sqrt{X_{21}} & \sqrt{X_{22}} \end{bmatrix}.$$

Then

$$X_{n+1} = \begin{bmatrix} ((X_n)_{11} + (X_n)_{12})/2 & (X_n)_{11} \# (X_n)_{12} \\ (X_n)_{21} \# (X_n)_{22} & ((X_n)_{21} + (X_n)_{22})/2 \end{bmatrix}$$

and the limit of X_n as $n \rightarrow \infty$ is

$$\begin{bmatrix} \mathbf{AG}(A, B) & \mathbf{AG}(A, B) \\ \mathbf{AG}(A, B) & \mathbf{AG}(A, B) \end{bmatrix}.$$

□

2 Archimedean double-mean process

If $a_{n+1} = m_1(a_n, b_n)$ and $b_{n+1} = m_2(a_{n+1}, b_n)$, then this is called an **Archimedean double-mean process** [6, 13]. Since $b_{n+1} = m_2(m_1(a_n, b_n), b_n)$, it is also a Gaussian process with means m_1 and $m_3(x, y) = m_2(m_1(x, y), y)$ where m_3 is non-symmetric in general. The proof of Theorem 1.2 can be easily modified to the Archimedean process.

We study the recurrence

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_{n+1}b_n} \quad (8)$$

which is in the next theorem. The proof can be found in [6] but we present it here for convenience. The functions

$$\operatorname{arcosh} x = \log(x + \sqrt{x^2 - 1}) \quad (x \geq 1)$$

and

$$\arccos x = -i \log(z + i\sqrt{1 - x^2}) \quad (0 < x < 1)$$

will appear.

Theorem 2.1. *If $a_0 = a$ and $b_0 = b$, then the sequences (a_n) , (b_n) defined in (8) are convergent and*

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = \begin{cases} \frac{\sqrt{b^2 - a^2}}{\arccos(a/b)}, & \text{if } 0 \leq a < b, \\ a, & \text{if } a = b, \\ \frac{\sqrt{a^2 - b^2}}{\operatorname{arcosh}(a/b)}, & \text{if } 0 < b < a. \end{cases}$$

Proof. Suppose $a \geq b$. Then by induction it follows that $a_n \geq a_{n+1} \geq b_{n+1} \geq b_n$ so that (a_n) is monotone decreasing and (b_n) is monotone increasing thus the sequences converge. If $\lim_{n \rightarrow \infty} a_n = \alpha$ and $\lim_{n \rightarrow \infty} b_n = \beta$, then by passing to the limit in the recurrence (8) we obtain that $\alpha = (\alpha + \beta)/2$ hence $\alpha = \beta$. Now denote

$$\Phi(x, y) := \frac{\sqrt{x^2 - y^2}}{\operatorname{arcosh}(x/y)}.$$

From L'Hospital's rule it follows $\Phi(x, x) = x$ for $x > 0$. Therefore, if we show that $\Phi(a_{n+1}, b_{n+1}) = \Phi(a_n, b_n)$ for every $n \in \mathbb{N}$, then passing to the limit will imply $\Phi(a, b) = \Phi(\lim_{n \rightarrow \infty} a_n, \lim_{n \rightarrow \infty} b_n) = \lim_{n \rightarrow \infty} a_n$, in other words, we apply the invariance principle for Φ . By simple calculation

$$\begin{aligned} \Phi(a_{n+1}, b_{n+1}) &= \frac{\sqrt{\left(\frac{a_n+b_n}{2}\right)^2 - \frac{a_n+b_n}{2}b_n}}{\operatorname{arcosh} \sqrt{\frac{a_n+b_n}{2b_n}}} = \frac{\sqrt{a_n^2 - b_n^2}}{2 \operatorname{arcosh} \sqrt{\frac{a_n+1}{b_n}}} \\ &= \frac{\sqrt{a_n^2 - b_n^2}}{\operatorname{arcosh} \frac{a_n}{b_n}} = \Phi(a_n, b_n) \end{aligned}$$

where we used the well-known identity $2 \operatorname{arcosh} \sqrt{\frac{x+1}{2}} = \operatorname{arcosh} x$.

The proof of the case $a \leq b$ is analogous. □

Example 2.2. A few applications appeared in the paper [11]. First

$$a_0 = \sqrt{x} = \frac{1+x}{2} \frac{2\sqrt{x}}{x+1} \quad \text{and} \quad b_0 = \frac{2x}{x+1} = \sqrt{x} \frac{2\sqrt{x}}{x+1},$$

then the limit is

$$\frac{x-1}{\log x} \frac{2\sqrt{x}}{x+1}.$$

Since we started with matrix monotone functions, this is matrix monotone as well. Next

$$a_0 = \frac{x-1}{\log x} \quad \text{and} \quad b_0 = \frac{x-1}{\log x} \frac{2\sqrt{x}}{1+x},$$

then the limit is the matrix monotone function

$$\left(\frac{x-1}{\log x}\right)^2 \frac{2}{1+x}.$$

The above function looks unusual but it corresponds to a monotone metric. This functions was conjectured in a paper of Morozova and Chentsov as use for quantum Fisher information and the matrix monotonicity was proved by Petz, see [11]. □

3 Logarithmic mean

The **logarithmic mean** of positive numbers a, b is defined as

$$L(a, b) = \frac{b-a}{\log b - \log a}$$

with representing function $f(x) = (x-1)/\log x$. Since

$$\frac{1}{L(a, b)} = \int_0^\infty \frac{dt}{(t+a)(t+b)} = \int_0^\infty \frac{dt}{(ta+b)(t+1)}, \quad (9)$$

we have for matrices

$$L(A, B)^{-1} = \int_0^\infty \frac{(tA+B)^{-1}}{t+1} dt. \quad (10)$$

Example 3.1. If we choose

$$a_0 = \frac{a+b}{2}, \quad b_0 = \sqrt{ab}$$

in the Archimedean double-mean process in Theorem 2.1, then the limit is the logarithmic mean $L(a, b)$. □

If a mean is determined by a matrix monotone function $f(x)$, then sometimes $f(x^p)^{1/p}$ gives another mean for some $p > 0$.

Proposition 3.2. *Let $0 < p \leq 2$ and suppose that $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is real analytic and has a holomorphic continuation to the sector $\{0 < \arg z < p\pi\}$ such that this sector is mapped into itself by f . Then $f(x^p)^{1/p}$ is matrix monotone.*

Proof. By Löwner's theorem (see [3]) $f(x^p)^{1/p}$ is matrix monotone if and only if it is real analytic and admits a holomorphic continuation to the open upper half-plane such that the open upper half-plane is mapped into itself by $f(x^p)^{1/p}$. By using complex logarithm we can define the function x^p holomorphically in the upper half-plane mapping the upper half-plane into the sector $\{0 < \arg z < p\pi\}$ thus we can define $f(x^p)$ holomorphically mapping the upper half-plane into this sector and so $f(x^p)^{1/p}$ mapping the upper half-plane into itself. \square

Since x^{-1} is matrix monotone decreasing we obtain the following property.

Proposition 3.3. *If $f(x^p)^{1/p}$ is matrix monotone, then $f(x^{-p})^{-1/p}$ is also matrix monotone.*

The matrix monotonicity of the limiting case $p = 0$ follows from the next proposition.

Proposition 3.4. *Let the continuously differentiable function $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be the representing function of a symmetric mean. Then $\lim_{p \rightarrow 0} f(x^p)^{1/p} = \sqrt{x}$.*

Proof. By the symmetry we have $f'(1) = 1/2$ so that L'Hospital's rule implies

$$\lim_{p \rightarrow 0} f(x^p)^{1/p} = \lim_{p \rightarrow 0} \exp\left(\frac{f(x^p)}{p}\right) = \lim_{p \rightarrow 0} \exp\left(\frac{f'(x^p) \log x}{1}\right) = \sqrt{x}.$$

\square

In view of the above results, we consider the following generalization of the logarithmic mean:

$$L_p(a, b) = \left(\frac{b^p - a^p}{\log b^p - \log a^p}\right)^{1/p} = \left(\frac{b^p - a^p}{p(\log b - \log a)}\right)^{1/p}$$

with representing function

$$f_p(x) = f(x^p)^{1/p} = \left(\frac{x^p - 1}{\log x^p}\right)^{1/p} \quad (11)$$

where $p \in \mathbb{R}$ and $f(x) = (x - 1)/\log x$ is the representing function of the logarithmic mean.

Some properties of L_p is discussed below.

Proposition 3.5. For every fixed $0 < x \neq 1$ the function $f_p(x)$ given by (11) is strictly monotone increasing in $p \in \mathbb{R}$.

Proof. By simple calculation we have

$$\frac{d}{dp} f_p(x) = \frac{1}{p^2} f_p(x) \left(x^p \frac{\log x^p}{x^p - 1} + \log \frac{\log x^p}{x^p - 1} - 1 \right).$$

Let

$$h(y) := y \frac{\log y}{y - 1} + \log \frac{\log y}{y - 1} - 1.$$

Clearly, $h(1) = 0$. We show that $h'(y) < 0$ for $0 < y < 1$ and $h'(y) > 0$ for $y > 1$ then $\frac{d}{dp} f_p(x) > 0$ follows for $p \neq 0$, $x \neq 1$ hence f_p is strictly increasing. Since

$$h'(y) = \left(\frac{(y-1)^2}{\log^2 y} - y \right) \frac{\log y}{(y-1)^2 y}$$

and the logarithmic mean is greater than the geometric mean, i.e.,

$$\frac{y-1}{\log y} \geq \sqrt{y} \quad (y \neq 1),$$

therefore, $h'(y) < 0$ for $0 < y < 1$ and $h'(y) > 0$ for $y > 1$. □

Theorem 3.6. The function $f_p: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ defined by (11) is matrix monotone for $-2 \leq p \leq 2$.

We need a technical lemma.

Lemma 3.7. For fixed $r > 0$, $0 < \varphi < \pi$ the function

$$q \mapsto \frac{r^q - \cos(q\varphi)}{\sin(q\varphi)}$$

is strictly monotone increasing in $[0, \pi/\varphi)$.

Proof. First note that for $q = 0$ by L'Hospital's rule we find that

$$\lim_{q \rightarrow 0} \frac{r^q - \cos(q\varphi)}{\sin(q\varphi)} = \frac{\log r}{\varphi}.$$

By simple calculation we obtain

$$\frac{d}{dq} \left(\frac{r^q - \cos(q\varphi)}{\sin(q\varphi)} \right) = \frac{r^q \log r \sin(q\varphi) - r^q \varphi \cos(q\varphi) + \varphi}{\sin^2(q\varphi)}.$$

We show that for $r > 0$, $0 < \varphi < \pi$ and $0 < q < \pi/\varphi$,

$$r^q \log r \sin(q\varphi) - r^q \varphi \cos(q\varphi) + \varphi > 0$$

which yields the monotonicity. Indeed, by the well-known inequality

$$\log x \geq 1 - \frac{1}{x} \quad (x > 0)$$

it follows that

$$\log r = \frac{1}{q} \log r^q \geq \frac{1}{q} \left(1 - \frac{1}{r^q}\right)$$

thus

$$\begin{aligned} & r^q \log r \sin(q\varphi) - r^q \varphi \cos(q\varphi) + \varphi \\ & \geq r^q \frac{1}{q} \left(1 - \frac{1}{r^q}\right) \sin(q\varphi) - r^q \varphi \cos(q\varphi) + \varphi \\ & = \frac{1}{q} (q\varphi - \sin(q\varphi) + r^q (\sin(q\varphi) - q\varphi \cos(q\varphi))) > 0 \end{aligned}$$

where the last estimate is due to the well-known inequalities $\sin x < x$ and $x \cos x < \sin x$ for $0 < x < \pi$. \square

Proof of Theorem 3.6. Due to Propositions 3.4 and 3.3 we may suppose $0 < p < 2$. By defining the complex logarithm as $\log z = \log |z| + i \arg z$ ($0 \leq \arg z < 2\pi$) we can extend f holomorphically to the upper half-plane. We show that f maps the sector $\{0 < \arg z < p\pi\}$ into itself then by Proposition 3.2 it follows that f_p is matrix monotone. Clearly, it suffices to verify that $0 < \arg f(z) \leq \arg z$ for all $z \in \mathbb{C} \setminus \mathbb{R}_0^+$.

Let $z = r(\cos \varphi + i \sin \varphi)$ and suppose first that $0 < \varphi < \pi$. Denote $\alpha := \arg(z - 1)$ and $\beta := \arg \log z$ then obviously $0 < \alpha, \beta < \pi$ and $\arg f(z) = \alpha - \beta$. We first show that $\alpha > \beta$. It is easily seen that

$$\cot \alpha = \frac{r \cos \varphi - 1}{r \sin \varphi}, \quad \cot \beta = \frac{\log r}{\varphi}.$$

By applying Lemma 3.7 with $q = 0$ and $q = 1$ it follows that

$$\frac{\frac{1}{r} - \cos \varphi}{\sin \varphi} > \frac{\log \frac{1}{r}}{\varphi}$$

thus $\cot \alpha < \cot \beta$ hence $\alpha > \beta$.

We verify $\alpha - \beta < \varphi$. The well-known trigonometric addition formula for the cotangent function yields

$$\cot(\alpha - \varphi) = \frac{1 + \cot \alpha \cot \varphi}{\cot \varphi - \cot \alpha} = \frac{r - \cos \varphi}{\sin \varphi}.$$

Lemma 3.7 with $q = 0$ and $q = 1$ implies that

$$\cot(\alpha - \varphi) = \frac{r - \cos \varphi}{\sin \varphi} > \frac{\log r}{\varphi} = \cot \beta$$

whence $\alpha - \varphi < \beta$.

Assume now $\pi \leq \varphi < 2\pi$ then $\pi \leq \alpha < \varphi < 2\pi$, $0 < \beta < \pi$ hence $\alpha > \beta$ and $\alpha - \beta < \varphi$. \square

Now we show that L_p can be obtained also as the limit of a Gaussian double mean iteration. Define the sequences (a_n) , (b_n) by the following recurrence:

$$\begin{aligned} a_0 &:= a, & b_0 &:= b, \\ a_{n+1} &:= \sqrt{a_n \left(\frac{a_n^{p/2} + b_n^{p/2}}{2} \right)^{2/p}}, & b_{n+1} &:= \sqrt{b_n \left(\frac{a_n^{p/2} + b_n^{p/2}}{2} \right)^{2/p}}. \end{aligned} \quad (12)$$

Theorem 3.8. *The sequences (a_n) , (b_n) converge to a common limit:*

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = \left(\frac{a^p - b^p}{p(\log a - \log b)} \right)^{1/p}.$$

Proof. The convergence to a common limit follows from the general theorem on Gaussian double mean iterations (analogously to the proof of Theorem 2.1) since the above recurrence can be written in the form

$$a_{n+1} = m(a_n, b_n), \quad b_{n+1} = m(b_n, a_n)$$

where

$$m(x, y) = G(x, H_{p/2}(x, y)),$$

G being the geometric mean and

$$H_{p/2}(x, y) = \left(\frac{x^{p/2} + y^{p/2}}{2} \right)^{2/p}$$

the power mean with exponent $p/2$. We apply the invariance principle. Denote

$$\Phi(x, y) = \left(\frac{y^p - x^p}{\log y^p - \log x^p} \right)^{1/p}.$$

Then $\Phi(x, x) = x$, on the other hand

$$\begin{aligned} \Phi(a_{n+1}, b_{n+1}) &= \left(\frac{b_n^{p/2} \frac{a_n^{p/2} + b_n^{p/2}}{2} - a_n^{p/2} \frac{a_n^{p/2} + b_n^{p/2}}{2}}{\frac{1}{2} p (\log b_n - \log a_n)} \right)^{1/p} \\ &= \left(\frac{b_n^p - a_n^p}{p(\log a_n - \log b_n)} \right)^{1/p} = \Phi(a_n, b_n). \end{aligned}$$

Thus $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = \Phi(\lim_{n \rightarrow \infty} a_n, \lim_{n \rightarrow \infty} b_n) = \Phi(a, b)$. \square

Below we prove that the function

$$g_p(x) = \left(\frac{x^p + 1}{2} \right)^{1/p}$$

is matrix monotone if and only if $-1 \leq p \leq 1$. Thus $H_{p/2}$ is a matrix mean if and only if $-2 \leq p \leq 2$. Therefore, the iteration (12) is also convergent for positive matrices if $-2 \leq p \leq 2$ and the limit is also a matrix mean so that we obtained a different proof for Theorem 3.6.

Proposition 3.9. *For every $t > 0$ the function*

$$x \mapsto \left(\frac{x^p + t}{2} \right)^{1/p}$$

is matrix monotone if and only if $-1 \leq p \leq 1$.

Proof. Since $\arg((z+t)/2) < \arg z$ in the upper half-plane, therefore, the function $x \mapsto (x+t)/2$ has a holomorphic continuation to the sector $\{0 < \arg z < p\pi\}$ mapping this sector into itself. So that by Proposition 3.2 the function $x \mapsto ((x^p+t)/2)^{1/p}$ is matrix monotone for $-1 \leq p \leq 1$.

On the other hand, $((x^p+t)/2)^{1/p} = t((y^p+1)/2)^{1/p}$ where $y = x/t^{1/p}$. The function $y \mapsto ((y^p+1)/2)^{1/p}$ is symmetric so it can be matrix monotone only if it is between the representing function of the arithmetic and geometric means thus $-1 \leq p \leq 1$. Hence $((x^p+t)/2)^{1/p}$ can be matrix monotone only if $-1 \leq p \leq 1$. \square

If $p = 1$, then the iteration (12) has the form

$$\begin{aligned} a_0 &:= a, & b_0 &:= b, \\ a_{n+1} &:= \frac{a_n + \sqrt{a_n b_n}}{2}, & b_{n+1} &:= \frac{b_n + \sqrt{a_n b_n}}{2} \end{aligned}$$

and the sequences (a_n) , (b_n) converge to $L(a, b)$. This algorithm can be written in a block matrix form.

Example 3.10. This example is an analogue of Example 1.3. For $A, B > 0$ let

$$X_0 := \begin{bmatrix} A & A \\ B & B \end{bmatrix}$$

and

$$X_{n+1} := \frac{1}{2}(X_n)_{11}^{1/2} \left(\sqrt[2]{(X_n)_{11}^{-1/2} X_n (X_n)_{11}^{-1/2}} \right)^2 (X_n)_{11}^{1/2}.$$

Then

$$X_{n+1} = \begin{bmatrix} ((X_n)_{11} + (X_n)_{11} \# (X_n)_{21})/2 & ((X_n)_{11} + (X_n)_{11} \# (X_n)_{21})/2 \\ ((X_n)_{21} + (X_n)_{11} \# (X_n)_{21})/2 & ((X_n)_{21} + (X_n)_{11} \# (X_n)_{21})/2 \end{bmatrix}$$

and the limit of X_n as $n \rightarrow \infty$ is

$$\begin{bmatrix} L(A, B) & L(A, B) \\ L(A, B) & L(A, B) \end{bmatrix}.$$

□

Example 3.11. A recent generalization of the arithmetic-geometric mean is $M_p(a, b)$ which is defined as

$$\frac{1}{M_p(a, b)} = c_p \int_0^\infty \frac{dt}{((a^p + t^p)(b^p + t^p))^{1/p}},$$

where $0 < p < \infty$ and

$$\frac{1}{c_p} = \int_0^\infty \frac{dt}{(t^p + 1)^{2/p}}$$

is for normalization [5]. Clearly, $M_1(a, b) = L(a, b)$ by (9), $M_2(a, b) = \mathbf{AG}(a, b)$ and $M_0(a, b) = G(a, b)$ by Proposition 3.4. For $0 \leq p \leq 1$, M_p makes also matrix mean since $(x^p + t^p)^{1/p}$ is a matrix monotone function due to Proposition 3.9. However, numerical computation shows that M_p is a matrix mean for $0 \leq p \leq 2$. □

Example 3.12. In 1975 K. Stolarsky introduced the mean

$$S_{p,q}(a, b) := \left(\frac{q(a^p - b^p)}{p(a^q - b^q)} \right)^{1/(p-q)},$$

with representing function

$$h_{p,q}(x) = \left(\frac{q(x^p - 1)}{p(x^q - 1)} \right)^{1/(p-q)} \quad (13)$$

see [14, 10]. We have

$$\lim_{q \rightarrow 0} S_{p,q}(a, b) = \left(\frac{a^p - b^p}{p(\log a - \log b)} \right)^{1/p} = L_p(a, b)$$

and

$$\lim_{q \rightarrow p} S_{p,q} = \left(\exp \left(\frac{a^p \log a^p - b^p \log b^p}{a^p - b^p} - 1 \right) \right)^{1/p} = I(a^p, b^p)^{1/p}$$

where I is the so-called identric mean with representing function

$$x \mapsto \exp \left(\frac{x \log x}{x - 1} - 1 \right).$$

Observe that in general $h_{p,q}(x) = h_{q,p}(x)$, $h_{-p,-q}(x) = h_{p,q}(x^{-1})^{-1}$ and

$$h_{p,q}(x) = \left(\left(\frac{x^p - 1}{\frac{q}{p}((x^p)^{q/p} - 1)} \right)^{1/(1-(q/p))} \right)^{1/p} = h_{1,q/p}(x^p)^{1/p}.$$

□

The matrix monotonicity of $h_{1,q}$ was studied in [2]. Now we show the matrix monotonicity of $h_{p,q}$ for certain values of p and q .

Theorem 3.13. *The function $h_{p,q}: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is matrix monotone if $-2 \leq p \leq 2$, $-1 \leq q \leq 1$ or symmetrically $-2 \leq q \leq 2$, $-1 \leq p \leq 1$.*

Proof. The case when $p = 0$ or $q = 0$ follows from Theorem 3.6 or by passing to the limit, further, the case $p = q$ follows by passing to the limit since the pointwise limit of matrix monotone functions is also matrix monotone. Moreover, since $h_{p,q}(x) = h_{q,p}(x)$, $h_{-p,-q}(x) = h_{p,q}(x^{-1})^{-1}$ and x^{-1} is matrix monotone decreasing we only have to consider the cases $0 < q < p \leq 2$ and $-1 \leq q < 0 < |q| < p \leq 2$. For these values of p and q by Löwner's theorem we have to show that the function $h_{p,q}$ has a holomorphic continuation to the upper half-plane mapping the upper half-plane into itself. The holomorphic continuation of the function $(z^p - 1)/(z^q - 1)$ is easily defined by using the complex logarithm $\log z = \log |z| + i \arg z$ ($0 \leq \arg z < 2\pi$). Below we show that for $0 < q < p \leq 2$, this function maps holomorphically the upper half-plane into the sector $\{0 < \arg z < (p-q)\pi\} \subset \mathbb{C} \setminus \mathbb{R}_0^+$. Therefore, the $1/(p-q)$ -th power of $(z^p - 1)/(z^q - 1)$ may be defined holomorphically and thus $h_{p,q}$ has a holomorphic continuation to a mapping of the upper half-plane into itself. The case $-1 \leq q < 0 < |q| < p \leq 2$ will follow from the previous one.

Assume first $0 < q < p \leq 2$ (hence $q \leq 1$). Let $z = r(\cos \varphi + i \sin \varphi)$ where $0 < \varphi < \pi$ and denote $\alpha := \arg(z^p - 1)$, $\beta := \arg(z^q - 1)$. If $p\varphi < \pi$, then clearly $0 < p\varphi < \alpha < \pi$, $0 < q\varphi < \beta < \pi$. It is easy to see that

$$\cot \alpha = \frac{r^p \cos(p\varphi) - 1}{r^p \sin(p\varphi)}, \quad \cot \beta = \frac{r^q \cos(q\varphi) - 1}{r^q \sin(q\varphi)}.$$

Lemma 3.7 implies that

$$\frac{\frac{1}{r^p} - \cos(p\varphi)}{\sin(p\varphi)} > \frac{\frac{1}{r^q} - \cos(q\varphi)}{\sin(q\varphi)}$$

thus $\cot \alpha < \cot \beta$ hence $\alpha > \beta$. Further, by using addition formula for the cotangent function we obtain

$$\cot(\alpha - p\varphi) = \frac{r^p - \cos(p\varphi)}{\sin(p\varphi)}, \quad \cot(\beta - q\varphi) = \frac{r^q - \cos q\varphi}{\sin q\varphi},$$

therefore, Lemma 3.7 implies $\cot(\alpha - p\varphi) > \cot(\beta - q\varphi)$ thus $\alpha - \beta < (p-q)\pi$. Otherwise $\alpha \geq \pi$ and then $\beta < \pi \leq \alpha$, on the other hand, $\alpha < p\varphi < 2\pi$ thus $\alpha - \beta < (p-q)\pi$. Whence

$$0 < \arg \left(\frac{z^p - 1}{z^q - 1} \right)^{\frac{1}{p-q}} = \frac{1}{p-q}(\alpha - \beta) < \pi.$$

In case $-1 \leq q < 0 < |q| < p \leq 2$ we have

$$\left(\frac{q(z^p - 1)}{p(z^q - 1)} \right)^{\frac{1}{p-q}} = \left(z^{|q|} \frac{|q|(z^p - 1)}{p(z^{|q|} - 1)} \right)^{\frac{1}{p+|q|}}.$$

Since, by the previous case,

$$0 < \arg \left(z^{|q|} \frac{|q|(z^p - 1)}{p(z^{|q|} - 1)} \right) < |q|\pi + (p - |q|)\pi,$$

it follows

$$0 < \arg h_{p,q}(z) < \frac{p}{p + |q|}\pi < \pi.$$

□

Numerical computations show that the Stolarsky function $h_{p,q}$ is not matrix monotone outside the intervals given in Theorem 3.13.

Remark 3.14. Observe that $S_{2p,p} = H_p$ thus Theorem 3.13 implies that the representing function of the power mean is matrix monotone for $-1 \leq p \leq 1$ which is in accordance with Proposition 3.9.

4 Complementary means

Let P and M be two means. We say that the mean N is complementary to M with respect to P if the invariance equation holds:

$$P(M(x, y), N(x, y)) = P(x, y) \quad (x, y \in \mathbb{R}^+).$$

We also say that P is (M, N) invariant. From the invariance principle it follows that if P is (M, N) invariant, then the limit of the Gaussian double mean iteration defined with M and N is P , and conversely. We study two particular cases, namely, when P is the arithmetic and geometric mean.

If $P(x, y) = A(x, y) = (x + y)/2$, then the invariance equation yields that $N(x, y) = 2A(x, y) - M(x, y)$. It is clear that if N is a mean for numbers, then $2A - M$ is also a mean for numbers, and conversely. However, if M is a symmetric matrix mean, then $M \leq A$ thus $2A - M \geq A$ so that $2A - M$ will be a symmetric matrix mean if and only if $M = N = A$. We obtained that in the symmetric matrix mean case the invariance equation holds with $P = A$ if and only if $M = N = A$. We remark that if $P = A$ the invariance equation is called the Matkowski-Sutô equation, see [8].

Now let $P(x, y) = G(x, y) = \sqrt{xy}$. Then $N(x, y) = xy/M(x, y)$. Observe that if M is a symmetric matrix mean with representing function f , then the representing function of N is $f(x^{-1})^{-1}$ which is matrix monotone thus N is a symmetric matrix mean (we note that N is called the adjoint of M in [9]). A particular example is $M = A$ then $N = H$. Moreover, L_p and L_{-p} are complementary, $S_{p,q}$ and $S_{-p,-q}$ are also complementary with respect to the geometric mean. In general, if $f(x^p)^{1/p}$ is matrix monotone, then the complementary mean with respect to the the geometric mean has representing function $f(x^{-p})^{-1/p}$.

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