NOTES

Lebesgue's Road to Antiderivatives

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The following theorem is well known:

THEOREM. *If $f : [a, b] \to \mathbb{R}$ is continuous, then f has antiderivatives.*

In most calculus courses and books, this is proved as part of the fundamental theorem of calculus, which states that, for a fixed $x_0 \in (a, b)$ and every x where f is continuous, the integral function $F(x) = \int_{x_0}^{x} f(t)\, dt$ is differentiable and satisfies $F'(x) = f(x)$. Since the statement of the theorem does not involve an integral, it is natural to ask whether the theorem can be proved without using the theory of definite integrals. In fact, Henri Lebesgue himself provided such a proof of this theorem in 1904–1905 [**1**, pp. 85–89; **2**]. In this paper, we revisit Lebesgue's proof.

Lebesgue's method is based on piecewise linear approximations of continuous functions. The existence of antiderivatives for these special types of functions is straightforward, and we can show that their limit also has antiderivatives. Remarkably, this approach also yields a proof of a mean-value-type inequality and the uniform convergence of antiderivatives of uniformly convergent sequences of continuous functions.

Both approaches rely on a theorem of E. Heine [**4**, pp. 143, 263], which states that continuous functions defined on bounded closed intervals are uniformly continuous. In the traditional approach, it is used to show that continuous functions are integrable on bounded closed intervals. In Lebesgue's approach, it provides piecewise linear approximations of continuous functions. However, this latter method, contrary to the traditional one, does not require the notion of supremum or infimum and oscillation. As Lebesgue wrote in [**2**]:

> "If one is limited to continuous functions in the whole course, then it [the traditional method] can be replaced by the following slightly different method which seems simpler for me."

In this note we recall Lebesgue's ideas of [**2**] in a somewhat modernized form (but keeping some of the original notation) and fill in some details according to his closing remarks, suggesting some modification of the arguments. Although Lebesgue did use the Mean Value Theorem in his proof (and also the notion of supremum, infimum, and oscillation), he noted that he did it only to shorten the presentation and it can be avoided easily. We shall accept Lebesgue's challenge and avoid using the Mean Value Theorem.

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Lebesgue's proof revisited

We assume, as Lebesgue did, that f is continuous on the closed interval $[a, b]$.

We first show the existence of antiderivatives for the class of piecewise linear functions. The general case will follow by approximations.

Piecewise linear functions have antiderivatives A linear function $f(x) = mx + n$ has an antiderivative $(m/2)x^2 + nx + K$, where K is any constant. Note that these are the only quadratic antiderivatives, since the coefficients of x^2 and x are uniquely determined by the derivative. (In fact, these are the only antiderivatives, but we do not need this result.)

Suppose now that f is piecewise linear, which means that $f(x) = m_i x + n_i$ in $[a_i, a_{i+1}]$ for $i = 0, \dots, p-1$ where $a = a_0 < a_1 < a_2 < \cdots < a_p = b$, and the following matching conditions hold (see FIGURE 1):

$$m_i a_{i+1} + n_i = m_{i+1} a_{i+1} + n_{i+1} \quad (i = 0, \dots, p-2).$$

For a piecewise linear function f, a piecewise quadratic antiderivative F can be constructed as follows. Let

$$F(x) = \frac{m_0}{2}x^2 + n_0 x - \frac{m_0}{2}a_0^2 - n_0 a_0 \quad (x \in [a_0, a_1])$$

and define (recursively) for $i = 1, \dots, p-1$,

$$F(x) = \frac{m_i}{2}x^2 + n_i x + F(a_i) - \frac{m_i}{2}a_i^2 - n_i a_i \quad (x \in (a_i, a_{i+1}]).$$

The constants $F(a_i) - (m_i/2)a_i^2 - n_i a_i$ are chosen so that F is continuous at $x = a_1, a_2, \dots, a_{p-1}$. The matching conditions ensure the differentiability of F at these points. It follows from the above construction that F is differentiable in (a, b) and $F' = f$. Further, $F(a) = 0$ also holds.

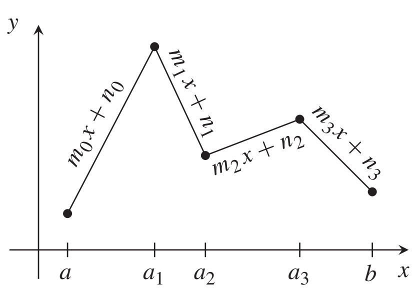

Figure 1 Piecewise linear function

It is also worth noticing that F is the unique piecewise quadratic antiderivative of the piecewise linear function f such that $F(a) = 0$, because the coefficients of the quadratic polynomials are uniquely determined by the coefficients of the derivative, the matching conditions, and the requirement that $F(a) = 0$. Consequently, a piecewise linear function admits a unique piecewise quadratic antiderivative up to an additive constant.

An inequality We now prove a simple inequality for piecewise quadratic functions. This will turn out to be a key result in the proof of the theorem.

LEMMA. *Let $f : [a, b] \to \mathbb{R}$ be a continuous piecewise linear function. Assume that $F : [a, b] \to \mathbb{R}$ is a continuous piecewise quadratic function such that F is differentiable in (a, b) and $F' = f$. Then*

$$(y - x) \min_{[x,y]} f \le F(y) - F(x) \le (y - x) \max_{[x,y]} f \tag{1}$$

for all $x, y \in [a, b]$ with $x < y$.

Proof. We may assume that f and F have the same form as above. First, suppose that x, y are in the same subinterval, $x, y \in [a_i, a_{i+1}]$, with $x < y$. Then

$$F(y) - F(x) = (y - x)\left(m_i \frac{y + x}{2} + n_i\right) = (y - x) f\left(\frac{y + x}{2}\right). \tag{2}$$

Since $(y + x)/2 \in [x, y]$, we conclude that $f(\frac{y+x}{2})$ is between $\min_{[x,y]} f$ and $\max_{[x,y]} f$. So (1) holds on a subinterval $[a_i, a_{i+1}]$.

In the general case, if $x \in [a_i, a_{i+1}]$, $x < y \in [a_j, a_{j+1}]$, then we may expand $F(y) - F(x)$ as a telescoping sum

$$F(y) - F(x) = (F(y) - F(a_j)) + (F(a_j) - F(a_{j-1})) + \cdots + (F(a_{i+1}) - F(x)).$$

By applying (1) on each subinterval, after summation (1) follows on the whole $[a, b]$. We note that the existence of the maximum and minimum in (1) does not require the Extreme Value Theorem. It is a consequence of the fact that f is continuous and piecewise monotone, so that the extrema are attained at the endpoints of the interval or at some point where f changes monotonicity. ■

Continuous functions Now that we have proved the theorem for a continuous piecewise linear function f, we next consider an arbitrary continuous function f. In this part of the proof of the theorem, inequalities (1) will play a central role.

Let now $f : [a, b] \to \mathbb{R}$ be an arbitrary continuous function. We show that f has antiderivatives. Since f is uniformly continuous by Heine's theorem, for every positive integer n there is $\delta_n > 0$, such that $|f(x) - f(y)| < 1/n$ whenever $|x - y| < \delta_n$. Take an arbitrary partition $a = a_0 < a_1 < a_2 < \cdots < a_r = b$ of the interval $[a, b]$ such that $a_{i+1} - a_i < \delta_n$ $(i = 0, \dots, r - 1)$, and define the piecewise linear function f_n to be equal to f at a_i and to be linear on $[a_i, a_{i+1}]$ for $i = 0, \dots, r - 1$ (see FIGURE 2).

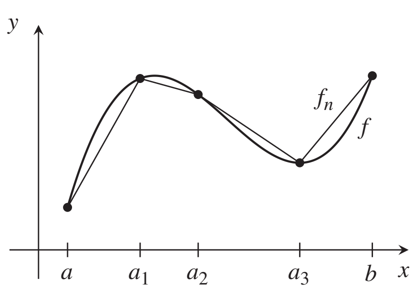

Figure 2 Piecewise linear approximation

If $x \in [a_i, a_{i+1}]$, then by the linearity, $f_n(x)$ is between the maximum and minimum of the values $f(a_i)$, $f(a_{i+1})$, and $f(x)$; hence, $f_n(x)$ and $f(x)$ differ by less than $1/n$.

So

$$|f_n(x) - f(x)| < 1/n \quad (x \in [a, b]). \tag{3}$$

Now consider the function $f_n - f_m$ for positive integers n, m. First note that by (3),

$$\begin{aligned} |f_n(x) - f_m(x)| &\le |f_n(x) - f(x)| + |f(x) - f_m(x)| \\ &< \frac{1}{n} + \frac{1}{m} \quad (x \in [a, b]). \end{aligned} \tag{4}$$

In addition, the piecewise linear functions f_n and f_m have piecewise quadratic antiderivatives F_n and F_m, respectively, with the property $F_n(a) = F_m(a) = 0$. Then $F_n - F_m$ is an antiderivative of $f_n - f_m$ and $F_n(a) - F_m(a) = 0$. Observe that the function $f_n - f_m$ is also piecewise linear, since the partitions corresponding to f_n and f_m divide $[a, b]$ into subintervals on which both f_n and f_m are linear, and so is their difference. Likewise, $F_n - F_m$ is piecewise quadratic. Therefore, we may apply (1) to the piecewise linear function $f_n - f_m$ and its piecewise quadratic antiderivative $F_n - F_m$, and we obtain for $x \in [a, b]$

$$|(F_n(x) - F_m(x)) - (F_n(a) - F_m(a))| \le |x - a| \max_{[a,x]} |f_n - f_m|.$$

Thus, $F_n(a) = F_m(a) = 0$ (in fact, only $F_n(a) = F_m(a)$ is necessary) and (4) imply

$$|F_n(x) - F_m(x)| \le |b - a| \left(\frac{1}{n} + \frac{1}{m}\right) \quad (x \in [a, b]). \tag{5}$$

This means that $(F_n(x))$ is a Cauchy sequence for every $x \in [a, b]$. Therefore, it is convergent, and $F_n(x) \to F(x)$ for some $F\colon [a, b] \to \mathbb{R}$.

We prove that F is differentiable in (a, b) and $F' = f$; moreover, $F'(a) = f(a)$ and $F'(b) = f(b)$ as one-sided derivatives. To this end, fix $x_0 \in [a, b]$ and let $\varepsilon > 0$. Since f is continuous at x_0, there is $\delta > 0$ such that

$$|f(x_0) - f(x)| < \varepsilon \quad \text{for} \quad |x - x_0| < \delta, x \in [a, b]. \tag{6}$$

We show that if $0 < |x_0 - x| < \delta$, $x \in [a, b]$, then

$$f(x_0) - \varepsilon \le \frac{F(x) - F(x_0)}{x - x_0} \le f(x_0) + \varepsilon, \tag{7}$$

which implies that $F'(x_0) = f(x_0)$. First, for $x \in [a, b]$, $x \ne x_0$, we have by (1) that

$$\min_{[x_0,x]} f_n \le \frac{F_n(x) - F_n(x_0)}{x - x_0} \le \max_{[x_0,x]} f_n, \tag{8}$$

where for $x < x_0$ the maximum and minimum are taken over $[x, x_0]$. Let $x_n^{\max} \in [x_0, x]$ be some point where $\max_{[x_0,x]} f_n$ is attained. Then, for $0 < |x - x_0| < \delta$, the inequalities (3) and (6) imply

$$\max_{[x_0,x]} f_n = f_n(x_n^{\max}) < f(x_n^{\max}) + \frac{1}{n} < f(x_0) + \varepsilon + \frac{1}{n}.$$

Analogously, we have for $0 < |x - x_0| < \delta$,

$$f(x_0) - \frac{1}{n} - \varepsilon < \min_{[x_0,x]} f_n.$$

So by (8) we obtain

$$f(x_0) - \frac{1}{n} - \varepsilon < \frac{F_n(x) - F_n(x_0)}{x - x_0} < f(x_0) + \frac{1}{n} + \varepsilon.$$

Now, (7) follows as $n \to \infty$.

There is more beneath

The Mean Value Theorem A key step in the proof was the lemma, along with inequality (1). This inequality would follow from the *Mean Value Theorem*, which says that if F is a continuous function on the closed interval $[a, b]$ and differentiable in (a, b), then

$$F(b) - F(a) = F'(\xi)(b - a) \quad \text{for some } \xi \in (a, b).$$

We did not rely on the Mean Value Theorem. Instead, we can deduce a variant of it from the lemma. Indeed, by (3) and (8),

$$\min_{[x_0,x]} f - \frac{1}{n} \le \min_{[x_0,x]} f_n \le \frac{F_n(x) - F_n(x_0)}{x - x_0} \le \max_{[x_0,x]} f_n \le \max_{[x_0,x]} f + \frac{1}{n}.$$

So as $n \to \infty$ it follows that

$$\min_{[x_0,x]} f \le \frac{F(x) - F(x_0)}{x - x_0} \le \max_{[x_0,x]} f. \tag{9}$$

We obtained that if $f\colon [a, b] \to \mathbb{R}$ is continuous, then it admits an antiderivative $F\colon [a, b] \to \mathbb{R}$ such that the mean value inequality (9) holds (where the existence of extrema follows from the Extreme Value Theorem). It is not difficult to verify in an elementary way, without using the Mean Value Theorem, that antiderivatives are unique up to an additive constant (see [**3**] for some references). Thus, inequality (9) holds for every antiderivative of f.

As we mentioned before, Lebesgue did use the Mean Value Theorem in his proof, but only to shorten the presentation. According to him, though the Mean Value Theorem is a simple and important result, its rigorous proof is rarely understood and in some courses it may be advantageous to replace it with a result like (9). Some decades later, the role of the Mean Value Theorem in the calculus curriculum was also intensively discussed; see [**3**] for a list of references.

Area and antiderivatives The equality (2) can be expressed graphically. Suppose that f is a positive continuous piecewise linear function, and choose $x = a_i$, $y = a_{i+1}$ in (2). Then

$$F(a_{i+1}) - F(a_i) = (a_{i+1} - a_i) f\left(\frac{a_i + a_{i+1}}{2}\right),$$

which means that the area of the trapezoid under the graph of the piecewise linear function f on the interval $[a_i, a_{i+1}]$ equals $F(a_{i+1}) - F(a_i)$ (see FIGURE 3). By summing the areas of the trapezoids from $i = 0$ to $i = p - 1$, we obtain that the area under the graph of the piecewise linear function f is $F(b) - F(a)$, where F is a continuous piecewise quadratic antiderivative of f. Since antiderivatives are unique up to an additive constant, it holds for every antiderivative of f.

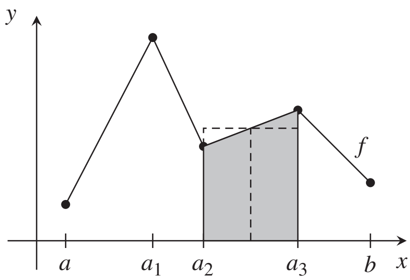

Figure 3 Area under a piecewise linear function

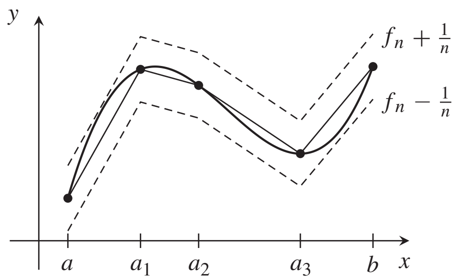

Figure 4 Area under a continuous function

We can extend this result to an arbitrary positive continuous function $f: [a, b] \to \mathbb{R}$. Choose a piecewise linear approximation f_n to f with the property (3). Then f is between the functions $f_n - \frac{1}{n}$ and $f_n + \frac{1}{n}$. If n is large enough, then $f_n - \frac{1}{n}$ is also positive; thus the area under the graph of f is between the area under $f_n - \frac{1}{n}$ and the area under $f_n + \frac{1}{n}$ (see FIGURE 4). Since $F_n(x) - \frac{x}{n}$ and $F_n(x) + \frac{x}{n}$ are antiderivatives of $f_n - \frac{1}{n}$ and $f_n + \frac{1}{n}$, respectively, it follows that

$$F_n(b) - F_n(a) - \frac{b-a}{n} \le \text{area under the graph of } f \le F_n(b) - F_n(a) + \frac{b-a}{n}.$$

Now, as $n \to \infty$ we obtain that

$$\text{area under the graph of } f = F(b) - F(a),$$

where $F: [a, b] \to \mathbb{R}$ is continuous and $F' = f$ in (a, b). This can be regarded as an "integral-free" version of the so-called second fundamental theorem of calculus. It can also be seen as a version of the Newton–Leibniz formula for a (positive) continuous function $f: [a, b] \to \mathbb{R}$, which states that $\int_a^b f(t)\,dt = F(b) - F(a)$ [**4**, p. 286]. In other words, we have proved the Riemann integrability of continuous functions, and the approach is different from the standard one.

Uniform convergence and antiderivatives Observe that inequalities (3) and (5) express the uniform convergence of (f_n) and (F_n), respectively. Consequently, as Lebesgue did in [**1**, p. 88], we proved much more.

THEOREM. *Let $f_n: [a, b] \to \mathbb{R}$ $(n = 1, 2 \ldots)$ be continuous functions with continuous antiderivatives $F_n: [a, b] \to \mathbb{R}$, such that $F_n(a) = F_1(a)$ $(n = 1, 2 \ldots)$. If $f_n \to f$ uniformly in $[a, b]$, then $F_n \to F$ uniformly in $[a, b]$, where $F: [a, b] \to \mathbb{R}$ is continuous and $F' = f$ in (a, b).*

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Summary The traditional way of proving the existence of antiderivatives of continuous functions is through the concept of definite integrals. In the years 1904–1905, H. Lebesgue provided an alternative proof of this result not relying on the theory of integrals. His method is based on piecewise linear approximations of continuous functions, which also yields the mean value inequality as a by-product. In this note we recall Lebesgue's ideas.