LEIBNIZ SEMINORMS IN PROBABILITY SPACES

ÁDÁM BESENYEI AND ZOLTÁN LÉKA

Abstract. In this paper we study the (strong) Leibniz property of centered moments of bounded random variables. We shall answer a question raised by M. Rieffel on the non-commutative standard deviation.

1. Introduction

We say that a seminorm L on a unital normed algebra $(\mathcal{A}, \|\cdot\|)$ is strongly Leibniz if (i) $L(1_{\mathcal{A}}) = 0$, (ii) the Leibniz property

$$L(ab) \leq \|a\|L(b) + \|b\|L(a)$$

holds for every $a, b \in \mathcal{A}$ and, furthermore, (iii) for every invertible a,

$$L(a^{-1}) \leq \|a^{-1}\|^2 L(a)$$

follows. Primary sources of strongly Leibniz seminorms are normed first-order differential calculi, see [8]. It is said that the couple (Ω, δ) is a normed first-order differential calculus over $\mathcal{A}$ if Ω is a normed bimodule over $\mathcal{A}$ and δ is a derivation from $\mathcal{A}$ to Ω. Now let us assume that Ω is acting boundedly over $\mathcal{A}$; that is, the inequalities

$$\|a\omega\| \leq \|\omega\|_\Omega \|a\| \quad \text{and} \quad \|\omega a\| \leq \|\omega\|_\Omega \|a\|$$

hold for every $\omega \in \Omega$ and for every $a \in \mathcal{A}$. From the derivation rule

$$\delta(ab) = \delta(a)b + a\delta(b),$$

the Leibniz property of the seminorm $L(a) = \|\delta(a)\|_\Omega$ simply follows. Furthermore, we clearly have that

$$\delta(a^{-1}) = -a^{-1}\delta(a)a^{-1},$$

whenever a is invertible, hence (iii) follows as well. For instance, if we choose a (real or complex) Banach space X and $\mathcal{B}(X)$ denotes the normed algebra of its bounded linear operators, practically, we can easily get a first-order differential calculus. Actually, with the choice of $\Omega = \mathcal{B}(X)$, which acts naturally over $\mathcal{B}(X)$ via the left and right multiplications, the commutator $\delta(A) = [D, A] = DA - AD$ for some fixed $D \in \mathcal{B}(X)$ defines the required calculus.

Consider a unital C^*-algebra $\mathcal{A}$ and denote $\mathcal{B}$ a C^*-subalgebra of $\mathcal{A}$ with a common unit. Rieffel pointed out in [7, Theorem] that the factor norm $\inf_{b\in\mathcal{B}} \|a-b\|$ obeys the strong Leibniz property, since it equals to a commutator norm. To get connection with the standard deviation, notice that K. Audenaert provided sharp estimate for different types of non-commutative (or quantum) deviations determined by matrices [1]. Not long ago Rieffel extended these results to C^*-algebras with a completely different approach [8]. His theorem reads as follows: for any $a \in \mathcal{A}$,

$$\max_{\omega\in\mathcal{S}(\mathcal{A})} \omega(|a - \omega(a)|^2)^{1/2} = \min_{\lambda\in\mathbb{C}} \|a - \lambda \mathbf{1}_{\mathcal{A}}\|,$$

2000 *Mathematics Subject Classification.* Primary 46L53, 60E15 ; Secondary 26A51, 60B99.
Key words and phrases. standard deviation, Leibniz seminorm, central moments, C^*-algebra.
This study was partially supported by the Hungarian NSRF (OTKA) grant no. K104206 .

where $\mathcal{S}(\mathcal{A})$ denotes the state space of $\mathcal{A}$; i.e. the set of positive linear functionals of $\mathcal{A}$ with norm 1. For a short proof of this theorem, exploiting the Birkhoff–James orthogonality in operator algebras, the reader might see [2]. The factor norm on the left-hand side above indicates that 'the largest standard deviation' is a strongly Leibniz seminorm. Surprisingly, the standard deviation itself is a strongly Leibniz seminorm. Precisely, whenever $\sigma_2^\omega(a) = \omega(|a-\omega(a)|^2)^{1/2}$, the seminorm σ_2^ω on $\mathcal{A}$ is strongly Leibniz if ω is tracial [8, Proposition 3.4]. Moreover, if one defines the non-commutative standard deviation by the formula

$$\tilde{\sigma}_2^\omega(a) = \omega(|a-\omega(a)|^2)^{1/2} \vee \omega(|a^*-\omega(a^*)|^2)^{1/2},$$

then $\tilde{\sigma}_2^\omega$ is strongly Leibniz for any $\omega \in \mathcal{S}(\mathcal{A})$, see [8, Theorem 3.5] (without assuming that ω is tracial). Quite recently, the equality

$$\max_{\omega\in\mathcal{S}(\mathcal{A})} \omega(|a-\omega(a)|^k)^{1/k} = 2B_k^{1/k} \min_{\lambda\in\mathbb{C}} \|a-\lambda \mathbf{1}_{\mathcal{A}}\|$$

was proved in [4] for the kth central moments of normal elements, where k is even and B_k denotes the largest kth centered moment of the Bernoulli distribution. From this result it follows that 'the largest kth moments' in commutative C^*-algebras are strongly Leibniz as well.

The aim of the paper is to study whether general or higher-ordered centered moments possess the (strong) Leibniz property in ordinary probability spaces, or not. In the next section we shall give a rough estimate of the centered moments of products of bounded random variables which gives back Rieffel's statement on the standard deviation. After that we shall present some scattered Leibniz-type result for different moments on different (discrete, general) probability spaces. We leave open the question whether all centered moments in general probability spaces define a strongly Leibniz seminorm. Lastly, in Section 3, we shall answer affirmatively Rieffel's question on the standard deviation in non-commutative probability spaces.

2. Leibniz seminorms in function spaces

In this section we shall study the Leibniz property and similar estimates in ordinary probability spaces. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. For any $f\colon \Omega \to \mathbb{C} \in L^\infty(\Omega,\mu)$ and $1 \le p < \infty$, let us define

$$\sigma_p(f;\mu) = \left(\int_\Omega \left| f - \int_\Omega f\, d\mu \right|^p d\mu\right)^{1/p}$$

and

$$\sigma_\infty(f;\mu) = \text{ess sup } \left| f - \int_\Omega f\, d\mu \right|.$$

If no confusion can arise, we simply use the notation $\sigma_p(f)$. Relying on [8], we know that the standard deviation is a strongly Leibniz seminorm; that is, the inequalities

$$\sigma_2(fg) \le \|g\|_\infty \sigma_2(f) + \|f\|_\infty \sigma_2(g)$$

for $f, g \in L^\infty(\Omega,\mu)$, and

$$\sigma_2(1/f) \le \|1/f\|_\infty^2 \sigma_2(f)$$

whenever $1/f \in L^\infty(\Omega,\mu)$ hold. For the non-commutative analogues of the result, see [8].

We begin with an observation which shows that one can reduce the problem of the strongly Leibniz property to that of the discrete uniform distributions.

Proposition 2.1. *Fix $1 \le p < \infty$. The following statements are equivalent:*

(i) *For any probability space $(\Omega, \mathcal{F}, \mu)$, σ_p is a strongly Leibniz seminorm on $L^\infty(\Omega,\mu)$.*

(ii) *For every $n \in \mathbb{Z}_+$, σ_p is a strongly Leibniz seminorm on ℓ_n^∞ endowed with the uniform distribution.*

Proof. Obviously, (i) implies (ii). To see the reverse implication, choose pairwise disjoint sets $S_k \in \mathcal{F}$ $(1 \le k \le n)$. As usual χ_{S_k} denotes the characteristic function of the set S_k. Let us consider the measurable simple functions $f_n = \sum_{k=1}^n a_k \chi_{S_k}$ and $g_n = \sum_{k=1}^n b_k \chi_{S_k}$ on Ω. Let us assume that $\bigcup_{k=1}^n S_k = \Omega$, so that the constants $\mu(S_k)$ define a probability measure μ_n on the set $\mathbb{Z}_n = \{1, \ldots, n\}$. Then for any $\varepsilon > 0$ we can readily find a probability measure $\nu_n = (p_1, \ldots, p_n)$ such that $p_i \in \mathbb{Q}$ $(1 \le i \le n)$ and the inequalities

$$|\sigma_p(f_n; \mu_n) - \sigma_p(f_n; \nu_n)| \le \varepsilon$$
$$|\sigma_p(g_n; \mu_n) - \sigma_p(g_n; \nu_n)| \le \varepsilon$$
$$|\sigma_p(f_n g_n; \mu_n) - \sigma_p(f_n g_n; \nu_n)| \le \varepsilon$$

hold. Now let us choose the integers m and r_i such that $p_i = r_i/m$ for every $1 \le i \le n$. Then the map

$$\Phi\colon (c_1, \ldots, c_n) \mapsto (\underbrace{c_1, \ldots, c_1}_{r_1}, \ldots, \underbrace{c_n, \ldots, c_n}_{r_n})$$

defines an isometric algebra homomorphism from ℓ_n^∞ into ℓ_m^∞. Let λ_m denote the uniform distribution on the set $\mathbb{Z}_m$. We clearly have, for instance, $\sigma_p(f_n; \nu_n) = \sigma_p(\Phi(f_n); \lambda_m)$, hence

$$\sigma_p(f_n g_n; \nu_n) \le \|f_n\|_\infty \sigma_p(g_n; \nu_n) + \|g_n\|_\infty \sigma_p(f_n; \nu_n)$$

follows as well. Since ε can be arbitrary small, we obtain that σ_p is a Leibniz seminorm on $\ell_n^\infty(\mu_n)$. Now if we choose sequences $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ of measurable simple functions such that $f_n \to f$ and $g_n \to g$ in L^p norm, furthermore, $\|f_n\|_\infty = \|f\|_\infty$ and $\|g_n\|_\infty = \|g\|_\infty$ hold for every n, we infer that σ_p has the Leibniz property. A very similar reasoning on the invertible elements gives that σ_p is actually strongly Leibniz on $L^\infty(\Omega, \mu)$. □

Despite of the above equivalence, in arbitrary measure spaces we do not know whether σ_p is strongly Leibniz or not. But later we will prove this property for σ_∞ in the real Banach space $L^\infty(\Omega, \mu; \mathbb{R})$ (see Theorem 2.6 below). Actually, the second part of the section deals with only real-valued functions. In the general situation, we have only a rough Leibniz-type estimate as we shall see below.

In any $L^p(\Omega, \mu)$ $(1 \le p \le \infty)$ space, the projection P is given by the map

$$f \mapsto \mathbb{E} f = \int_\Omega f \, d\mu.$$

Then we are able to prove a slight generalization of Rieffel's statement [8, Proposition 3.4] in probability spaces.

Proposition 2.2. *For any $1 \le p \le \infty$ and $f, g \in L^\infty(\Omega, \mu)$, we have that*

$$\frac{2}{\|I - P\|_p + 1} \|fg - \mathbb{E}(fg)\|_p \le \|g\|_\infty \|f - \mathbb{E} f\|_p + \|f\|_\infty \|g - \mathbb{E} g\|_p.$$

Proof. First, note that $\|I - P\|_p \ge 1$ (except for the trivial case $I = P$). Hence, without loss of generality, we can assume that

$$\|fg - \mathbb{E}(fg)\|_p \ge \max(\|f\|_\infty \|g - \mathbb{E} g\|_p, \|g\|_\infty \|f - \mathbb{E} f\|_p),$$

otherwise the proof is done. Obviously,

$$\|f(g - \mathbb{E} g) - \mathbb{E}(f(g - \mathbb{E} g))\|_p \le \|I - P\|_p \|f(g - \mathbb{E} g)\|_p.$$

From the reversed triangle inequality we obtain that

$$\|fg - \mathbb{E}(fg)\|_p - \|\mathbb{E}f\mathbb{E}g - f\mathbb{E}g\|_p \leq \|f(g-\mathbb{E}g) - \mathbb{E}(f(g-\mathbb{E}g))\|_p,$$

which implies that

$$\|fg - \mathbb{E}(fg)\|_p \leq \|I-P\|_p\|f\|_\infty\|g-\mathbb{E}g\|_p + \|g\|_\infty\|f-\mathbb{E}f\|_p.$$

Changing the variables f, g and summing up the inequalities, we get the statement of the proposition. □

Remark 2.3. One can find a non-trivial upper estimate of the constant $\|I-P\|_p$. For instance, if $\Omega = \{1,\dots,n\}$ and μ is the uniform distribution on Ω, from the definition of the matrix p-norms one can easily see that $\|I-P\|_1 = \|I-P\|_\infty = 2 - \frac{2}{n}$ and $\|I-P\|_2 = 1$. As another example, let us consider the Banach spaces $L^p[0,1]$ endowed with the Lebesgue measure. Then a simple calculation shows that $\|I-P\|_1 = \|I-P\|_\infty = 2$. Moreover, $I-P$ is clearly an orthogonal projection in $L^2[0,1]$; that is, $\|I-P\|_2 = 1$. Now a straightforward application of the Riesz–Thorin interpolation theorem gives that (see [6])

$$\|I-P\|_p \leq 2^{|1-\frac{1}{2p}|}.$$

The projection $I-P$ is actually the minimal projection to the hyperlane $X_p = \{f \in L^p[0,1] : \mathbb{E}f = 0\}$; i.e. it has the minimal norm among the projections of range X_p. C. Franchetti showed in his paper [3] that

$$\|I-P\|_p = \max_{0\leq x\leq 1}(x^{p-1} + (1-x)^{p-1})^{1/p}(x^{q-1} + (1-x)^{q-1})^{1/q},$$

where $1/p + 1/q = 1$.

Remark 2.4. One can apply a derivation approach mentioned in the Introduction to obtain Leibniz-type estimates of the moments of invertible functions. To do this, let us renorm the space $L^p(\Omega,\mu)$, $2 \leq p < \infty$, so that

$$\|x\|_{p,\vee} := |\mathbb{E}x| + \|x - \mathbb{E}x\|_p.$$

Let X denote the renormed space. Define the multiplication operator $M_f\colon x \mapsto fx$ and the derivation $\delta(M_f) = [P, M_f] = PM_f - M_fP$. A straightforward calculation yields that

$$\|M_fx\|_{p,\vee} \leq \|f\|_\infty\|x\|_p + \|I-P\|_p\|f\|_\infty\|x\|_p \leq (1 + \|I-P\|_p)\|f\|_\infty\|x\|_{p,\vee};$$

that is, $\|M_f\| \leq (1+\|I-P\|_p)\|f\|_\infty$. Moreover, $\delta(M_f)\mathbb{E}x = \mathbb{E}x(\mathbb{E}f - f) \in (I-P)X$, thus $\|\delta(M_f)_{|PX}\| = \sigma_p(f)$. On the other hand, $\delta(M_f)(x - \mathbb{E}x) = \mathbb{E}(fx) - \mathbb{E}f\mathbb{E}x = \mathbb{E}((f-\mathbb{E}f)(x-\mathbb{E}x))$. From Hölder's inequality we get that

$$\|\delta(M_f)(x-\mathbb{E}x)\|_{p,\vee} \leq \|f-\mathbb{E}f\|_q\|x-\mathbb{E}x\|_p \qquad (1/q + 1/p = 1)$$

hence $\|\delta(M_f)_{|(I-P)X}\| \leq \sigma_p(f)$ follows. Since the operator $\delta(M_f)$ interchanges the subspaces PX and $(I-P)X$, we have

$$\|\delta(M_f)\| = \sigma_p(f).$$

An application of the derivation rules tells us that

$$\sigma_p(1/f) \leq (1 + \|I-P\|_p)^2\|1/f\|_\infty^2\sigma_p(f)$$

holds whenever $1/f \in L^\infty(\Omega,\mu)$.

For any $1 \leq p \leq \infty$, we can get a different estimate from the equality

$$(I-P)M_{1/f}(I-P)f = (1/f - \mathbb{E}(1/f))\mathbb{E}f.$$

Hence we conclude that

$$|\mathbb{E}f|\sigma_p(1/f) \leq \|I-P\|_p\|1/f\|_\infty\sigma_p(f).$$

Much of the rest of the section is devoted to a study of the optimality of the above proposition. We begin with the following observation.

Proposition 2.5. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. For any real-valued f and $x \in L^\infty(\Omega, \mu)$, the inequality*

$$\|f\mathbb{E}x - \mathbb{E}(fx)\|_\infty \le \|x\|_\infty \|f - \mathbb{E}f\|_\infty$$

holds.

Proof. Without loss of generality, we can assume that $\mathbb{E}f = 0$ holds and $\|f\|_\infty = 1$. Note that the function $f \mapsto f\mathbb{E}x - \mathbb{E}(fx)$ is convex on the weak-$*$ compact, convex set

$$L_0^\infty(\Omega) := \{f \in L^\infty(\Omega, \mu) \colon \|f\|_\infty \le 1 \text{ and } \mathbb{E}f = 0\} \subseteq (L^1(\Omega, \mu))^*.$$

Hence, from the Krein–Milman theorem, it is enough to prove the statement if f is an extreme point of $L_0^\infty(\Omega)$. We claim that the extreme points of $L_0^\infty(\Omega)$ are the functions with essential range $\{-1, 1, c\}$ for some $-1 < c < 1$, ($\mu(\{f = c\}) = 0$ might be possible) and

$$\mathbb{E}f = \mu(\{f = 1\}) - \mu(\{f = -1\}) + c\mu(\{f = c\}) = 0. \tag{2.1}$$

Let us choose a measurable subset A of Ω such that $\|f\chi_A\|_\infty \le 1 - \varepsilon < 1$. If μ is non-atomic (A is not a singleton), we can find a function $g \in L_0^\infty(\Omega)$ satisfying $\|g\|_\infty \le \varepsilon$ and $g = 0$ a.e. on $\Omega \setminus A$. Since

$$f = \frac{1}{2}(f + g) + \frac{1}{2}(f - g),$$

f is an extreme point if and only if $\mu(A) = 0$. When μ is atomic, the set A might be a singleton, hence our claim follows.

Now let f be an extreme point of $L_0^\infty(\Omega)$. Obviously, $\|f - \mathbb{E}f\|_\infty = 1$. Furthermore, we have

$$\begin{aligned}
\|f\mathbb{E}x - \mathbb{E}(fx)\|_\infty &= \max(|\mathbb{E}x - \mathbb{E}(fx)|, |\mathbb{E}x + \mathbb{E}(fx)|, |c\mathbb{E}x - \mathbb{E}(fx)|) \\
&= \max(|\mathbb{E}(x(1 - f))|, |\mathbb{E}(x(1 + f))|, |\mathbb{E}(x(c - f))|) \\
&\le \|x\|_\infty \max(\|1 - f\|_1, \|1 + f\|_1, \|c - f\|_1).
\end{aligned}$$

It remains to show that $\max(\|1 - f\|_1, \|1 + f\|_1, \|c - f\|_1) = 1$. Clearly, from (2.1)

$$\begin{aligned}
\|1 - f\|_1 &= 2\mu(\{f = -1\}) + |1 - c|\mu(\{f = c\}) \\
&= 1 - \mu(\{f = 1\}) + \mu(\{f = -1\}) - c\mu(\{f = c\}) = 1.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|1 + f\|_1 &= 2\mu(\{f = 1\}) + |1 + c|\mu(\{f = c\}) \\
&= 1 + \mu(\{f = 1\}) - \mu(\{f = -1\}) + c\mu(\{f = c\}) = 1,
\end{aligned}$$

and lastly we infer that

$$\begin{aligned}
\|c - f\|_1 &= |c - 1|\mu(\{f = 1\}) + |c + 1|\mu(\{f = -1\}) \\
&= \mu(\{f = 1\}) + \mu(\{f = -1\}) + c^2\mu(\{f = c\}) \le 1.
\end{aligned}$$

The proof is complete. □

For the real Banach space $L^\infty(\Omega, \mu; \mathbb{R})$, we can simply prove that the seminorm σ_∞ is strongly Leibniz as we have seen before for the standard deviation.

Theorem 2.6. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. For the real Banach space $L^\infty(\Omega, \mu; \mathbb{R})$,*

$$\sigma_\infty(f) = \|f - \mathbb{E}f\|_\infty$$

is a strongly Leibniz seminorm.

Proof. From Proposition 2.5, it follows that

$$\begin{aligned}\|fg - \mathbb{E}(fg)\|_p &= \|f(g - \mathbb{E}g) + (f\mathbb{E}g - \mathbb{E}(fg))\|_p \\ &\le \|f\|_\infty \|g - \mathbb{E}g\|_p + \|g\|_\infty \|f - \mathbb{E}f\|_p,\end{aligned}$$

and

$$\left\|\frac{1}{f} - \mathbb{E}\frac{1}{f}\right\|_p = \left\|\frac{1}{f}\left(\mathbb{E}\left(f\cdot\frac{1}{f}\right) - f\mathbb{E}\frac{1}{f}\right)\right\|_p \le \left\|\frac{1}{f}\right\|_\infty \cdot \left\|\frac{1}{f}\right\|_\infty \cdot \|f - \mathbb{E}f\|_p,$$

which is what we intended to have. □

Regarding the case of the uniform distributions seen above in Proposition 2.1, we are able to prove the analogue of Proposition 2.5 in very particular cases. Let λ_n stand for the uniform distribution on $\mathbb{Z}_n$.

Proposition 2.7. *Fix $1 \le n \le 4$. For $1 \le p < \infty$, and any real-valued $f, x \in \ell_n^\infty(\lambda_n)$, we have*

$$\|f\mathbb{E}x - \mathbb{E}(fx)\|_p \le \|x\|_\infty \|f - \mathbb{E}f\|_p.$$

Proof. First note that the case $\Omega = \mathbb{Z}_1$ is trivial. On the other hand, in case of $\Omega = \mathbb{Z}_2$, one can have arbitrary distribution. Indeed, let $\mu(1) = p_1$ and $\mu(2) = p_2 = 1 - p_1$. Then by simple calculation we obtain

$$f - \mathbb{E}f = (f_1 - f_2)\cdot(p_2, -p_1)$$

and

$$f\mathbb{E}x - \mathbb{E}(fx) = (f_1 - f_2)\cdot(p_2x_2, -p_1x_1)$$

so the desired inequality follows immediately.

To prove the remaining cases $\Omega = \mathbb{Z}_3$ and $\Omega = \mathbb{Z}_4$, let us rescale the inequality and assume that $\|x\|_\infty = 1$. Notice that the function

$$x \mapsto \|f\mathbb{E}x - \mathbb{E}(fx)\|_p$$

is convex on the closed unit ball $\{x \in L^\infty(\Omega,\mu) : \|x\|_\infty \le 1\}$, therefore it suffices to check the inequality only for its extreme points.

First, we turn to the case $\Omega = \mathbb{Z}_3$. Clearly, for $x = (1,1,1)$ even equality holds, so after possible rearrangement and multiplication by constants we may assume that $x = (1,1,-1)$. Then

$$f - \mathbb{E}f = \frac{1}{3}(2f_1 - f_2 - f_3, -f_1 + 2f_2 - f_3, -f_1 - f_2 + 2f_3)$$

and

$$f\mathbb{E}x - \mathbb{E}(fx) = \frac{1}{3}(f_3 - f_1, f_3 - f_2, 2f_3 - f_1 - f_2).$$

By using the notation $a_1 = 2f_1 - f_2 - f_3$ and $a_2 = 2f_2 - f_1 - f_3$, the inequality reduces to the form

$$\left|\frac{2a_1 + a_2}{3}\right|^p + \left|\frac{a_1 + 2a_2}{3}\right|^p \le |a_1|^p + |a_2|^p,$$

which is obviously true from the convexity of the function $t \mapsto |t|^p$.

Next, let $\Omega = \mathbb{Z}_4$. By symmetry arguments we can assume that $x = (1,1,1,-1)$ or $x = (1,1,-1,-1)$. Set $x = (1,1,1,-1)$. A simple calculation implies that

$$f - \mathbb{E}f = \frac{1}{4}(a_1, a_2, a_3, a_4),$$

where

$$a_j = 3f_j - \sum_{i \ne j} f_i.$$

Moreover,

$$f\mathbb{E}x - \mathbb{E}(fx) = \frac{1}{4}\begin{pmatrix} +f_1 - f_2 - f_3 + f_4 \\ -f_1 + f_2 - f_3 + f_4 \\ -f_1 - f_2 + f_3 + f_4 \\ -f_1 - f_2 - f_3 + 3f_4 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} -a_2 - a_3 \\ -a_1 - a_3 \\ -a_1 - a_2 \\ 2a_4 \end{pmatrix}.$$

Therefore, it is enough to check that

$$\left|\frac{a_2 + a_3}{2}\right|^p + \left|\frac{a_1 + a_3}{2}\right|^p + \left|\frac{a_1 + a_2}{2}\right|^p \le |a_1|^p + |a_2|^p + |a_3|^p,$$

which follows again by the convexity of the function $t \mapsto |t|^p$.

Lastly, consider the remaining case $x = (1, 1, -1, -1)$. Then

$$f\mathbb{E}x - \mathbb{E}(fx) = \frac{1}{4}\begin{pmatrix} -f_1 - f_2 + f_3 + f_4 \\ -f_1 - f_2 + f_3 + f_4 \\ -f_1 - f_2 + f_3 + f_4 \\ -f_1 - f_2 + f_3 + f_4 \end{pmatrix} = \frac{1}{16}\begin{pmatrix} -a_1 - a_2 + a_3 + a_4 \\ -a_1 - a_2 + a_3 + a_4 \\ -a_1 - a_2 + a_3 + a_4 \\ -a_1 - a_2 + a_3 + a_4 \end{pmatrix}.$$

Since

$$4\left|\frac{-a_1 - a_2 + a_3 + a_4}{4}\right|^p \le |a_1|^p + |a_2|^p + |a_3|^p + |a_4|^p,$$

by a convexity argument as seen before, we get the statement of the proposition. □

Example 2.8. The statement of Proposition 2.7 does not hold in general. Let $n \ge 5$ and $p = 1$, for instance. Let $x = (1, \dots, 1, -1)$ and $f = (1, 0, \dots, 0, -1)$ in $\ell_n^\infty(\lambda_n)$. Obviously, $\mathbb{E}f = 0$, $\mathbb{E}x = 1 - \frac{2}{n}$, $\mathbb{E}(fx) = \frac{2}{n}$, $\|x\|_\infty = 1$, furthermore,

$$\|f - \mathbb{E}f\|_1 = \frac{2}{n},$$

and

$$\|f\mathbb{E}x - \mathbb{E}(fx)\|_1 = \left\|\left(1 - \frac{4}{n}, -\frac{2}{n}, \dots, -\frac{2}{n}, -1\right)\right\|_1 = \frac{4n - 8}{n^2} \qquad (n \ge 5).$$

Thus

$$\|f\mathbb{E}x - \mathbb{E}(fx)\|_1 = \left(2 - \frac{4}{n}\right)\|f - \mathbb{E}f\|_1 > \|f - \mathbb{E}f\|_1.$$

Example 2.9. In the case of non-uniform distributions, the inequality of Proposition 2.7 is not true even on $\Omega = \{1, 2, 3\}$. To see this, define the measure $\mu(1) = \frac{1}{8}$, $\mu(2) = \frac{3}{4}$, $\mu(3) = \frac{1}{8}$ and consider $f = (1, 0, -1)$ and $x = (1, 1, -1)$. Then $\mathbb{E}f = 0$, $\mathbb{E}x = \frac{3}{4}$, $\mathbb{E}(fx) = \frac{1}{4}$, and

$$\|f - \mathbb{E}f\|_1 = \frac{1}{4},$$

while

$$\|f\mathbb{E}x - \mathbb{E}(fx)\|_1 = \left\|\left(\frac{1}{2}, -\frac{1}{4}, -1\right)\right\|_1 = \frac{3}{8}.$$

As we have seen before in the proof of Theorem 2.6, we can infer the next statement on discrete measure spaces.

Corollary 2.10. *For $1 \le n \le 4$ and $1 \le p < \infty$, the seminorm σ_p is strongly Leibniz on the real ℓ_n^∞ endowed with uniform distribution.*

Surprisingly, we cannot prove or disprove the last statement on measure spaces which contain more than 4 atoms. Computer simulations suggest us that Corollary 2.10 might be true for any n which would imply that σ_p is a strongly Leibniz seminorm for every $1 \le p < \infty$ (see Proposition 2.1). Now we have only a very few particular results on general measure spaces. Denote λ_n the uniform distribution on the set $\mathbb{Z}_n$, as usual.

Proposition 2.11. *Let $1 \le p < \infty$ and $f,\, g \in \ell_n^\infty(\lambda_n)$ be such that the coordinates of f, g and fg have the same order. Then*

$$\|fg - \mathbb{E}(fg)\|_p \le \|g\|_\infty \|f - \mathbb{E}f\|_p + \|f\|_\infty \|g - \mathbb{E}g\|_p$$

holds.

Proof. We use the fact that the ℓ^p norm with uniform distribution and $1 \le p \le \infty$ is a Schur-convex function [5, Ch. 3 Example I.1]. Therefore, it suffices to prove that the vector $fg - \mathbb{E}(fg)$ is majorized by $\|f\|_\infty(g - \mathbb{E}g) + \|g\|_\infty(f - \mathbb{E}f)$. To see this, we may assume without loss of generality that $f_1 \ge f_2 \ge \cdots \ge f_n$, thus we also have $g_1 \ge g_2 \ge \cdots \ge g_n$ and $f_1 g_1 \ge f_2 g_2 \ge \cdots \ge f_n g_n$. Then we have to verify that

$$\sum_{j=1}^k (f_j g_j - \mathbb{E}(fg)) \le \sum_{j=1}^k \left(\|g\|_\infty (f_j - \mathbb{E}f) + \|f\|_\infty (g_j - \mathbb{E}g)\right),$$

for all $1 \le k \le n-1$, and equality holds when $k = n$. The latter equality is obvious because both sides are zero if $k = n$. In the remainder of the proof, a simple calculation gives that

$$n\left(\sum_{j=1}^k (f_j - \mathbb{E}f)\right) = (n-k)\sum_{j=1}^k f_j - k\sum_{j=k+1}^n f_j = \sum_{j=1}^k \sum_{i=k+1}^n (f_j - f_i)$$

and analogously

$$\begin{aligned} n\left(\sum_{j=1}^k (f_j g_j - \mathbb{E}(fg))\right) &= \sum_{j=1}^k \sum_{i=k+1}^n (f_j g_j - f_i g_i) \\ &= \sum_{j=1}^k \sum_{i=k+1}^n \left(f_j(g_j - g_i) + g_i(f_j - f_i)\right). \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} &\sum_{j=1}^k \left(\|f\|_\infty (g_j - \mathbb{E}g) + \|g\|_\infty (f_j - \mathbb{E}f) - (f_j g_j - \mathbb{E}(fg))\right) \\ &= \frac{1}{n}\left(\sum_{j=1}^k \sum_{i=k+1}^n (g_j - g_i)(\|f\|_\infty - f_j) + (f_j - f_i)(\|g\|_\infty - g_i)\right) \ge 0. \end{aligned}$$

□

Analogously to the proof of Proposition 2.1, we readily obtain the following corollaries.

Corollary 2.12. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $1 \le p < \infty$. For any non-negative $f \in L^\infty(\Omega, \mu)$,*

$$\|f^2 - \mathbb{E}f^2\|_p \le 2\|f\|_\infty \|f - \mathbb{E}f\|_p.$$

Corollary 2.13. *Let $1 \le p < \infty$ and μ be a probability measure on the interval $[0,1]$. For any non-negative, bounded and monotone increasing (or decreasing) functions f and g, we have*

$$\|fg - \mathbb{E}fg\|_p \le \|g\|_\infty \|f - \mathbb{E}f\|_p + \|f\|_\infty \|g - \mathbb{E}g\|_p.$$

3. Standard deviation in C^*-algebras

In this section we shall complete Rieffel's argument on the standard deviation in non-commutative probability spaces. Let $\mathcal{A}$ be a unital C^*-algebra and denote ω any faithful state of it. Denote $L^2(\mathcal{A},\omega)$ the GNS Hilbert space obtained by completing $\mathcal{A}$ for the inner product $\langle a,b\rangle = \omega(b^*a)$, as usual. Obviously, every $a \in \mathcal{A}$ has a natural representation; i.e. the left-regular representation L_a, in the operator algebra of $L^2(\mathcal{A},\omega)$. Consider now the projection (or Dirac operator) $E\colon a \mapsto \omega(a)\mathbf{1}_{\mathcal{A}}$ on $L^2(\mathcal{A},\omega)$. Direct calculations for the norm of the commutator $\delta(L_a) = [E, L_a] = EL_a - L_aE$ give that

$$\|\delta(L_a)\| = \omega(|a-\omega(a)|^2)^{1/2} \vee \omega(|a^*-\omega(a^*)|^2)^{1/2}.$$

Thus it immediately follows that Rieffel's non-commutative standard deviation is a strongly Leibniz $*$-seminorm, see [8, Theorem 3.7]. Moreover, an application of the 'independent copies trick' in C^*-algebras gives that

$$\sigma_2^{\omega}(a) := \omega(|a-\omega(a)|^2)$$

is strongly Leibniz as well if one assumes that ω is tracial [8, Proposition 3.6]. Actually, the 'strong' part of the statement requires only the tracial assumption. Computer simulations for matrices indicate that σ_2^{ω} might be strongly Leibniz for any state ω but the question remained open in [8]. Now we shall provide the affirmative answer by means of an elementary argument.

Pick a faithful state ω of $\mathcal{A}$. Let $\|a\|_2 = \omega(|a|^2)^{1/2}$ denote the norm on $L^2(\mathcal{A},\omega)$. We begin with

Lemma 3.1. *For any a and $x \in \mathcal{A}$,*

$$\|\omega(x)a - \omega(xa)\|_2 \le \|x\|\|a-\omega(a)\|_2.$$

Proof. There is no loss of generality in assuming that $\omega(a) = 0$. Denote E the orthogonal projection from $L^2(\mathcal{A},\omega)$ onto its subspace $\mathbb{C}\mathbf{1}_{\mathcal{A}}$. Then

$$\|\omega(x)a - \omega(xa)\|_2 = \|\omega(x)(I-E)a - E\omega(xa)\|_2 = \|\omega(x)a\|_2 + |\omega(xa)|.$$

Notice that the Cauchy–Schwarz inequality readily gives that

$$|\omega(xa)| = |\omega((x-\omega(x))a)| \le \|a\|_2\|x^* - \omega(x^*)\|_2.$$

Hence

$$\begin{aligned}\|\omega(x)a - \omega(xa)\|_2 &= \|\omega(x)a\|_2 + |\omega(xa)| \\ &\le |\omega(x^*)|\|a\|_2 + \|a\|_2\|x^*-\omega(x^*)\|_2 \\ &= \|x^*\|_2\|a\|_2 \\ &\le \|x^*\|\|a\|_2 \\ &= \|x\|\|a\|_2,\end{aligned}$$

and the proof is finished. □

Now the main theorem of the section reads as follows.

Theorem 3.2. *For any invertible $a \in \mathcal{A}$, the inequality*

$$\|a^{-1} - \omega(a^{-1})\|_2 \le \|a^{-1}\|^2\|a-\omega(a)\|_2$$

holds.

Proof. We clearly have that

$$\|xa\|_2 \le \|x\|\|a\|_2$$

for any $x \in \mathcal{A}$. In fact,

$$\omega(|xa|^2) = \omega(a^*|x|^2a) \le \omega(a^*\|x\|^2a) = \|x\|^2\omega(|a|^2).$$

Combining the previous inequality with Lemma 3.1, it follows that

$$\begin{aligned}\|a^{-1}-\omega(a^{-1})\|_2 &= \|a^{-1}(\omega(a^{-1}a)-\omega(a^{-1})a))\|_2\\ &= \|a^{-1}(\omega(a^{-1}a)-\omega(a^{-1})a))\|_2\\ &\le \|a^{-1}\|\|\omega(a^{-1}a)-\omega(a^{-1})a)\|_2\\ &\le \|a^{-1}\|^2\|a-\omega(a)\|_2,\end{aligned}$$

and the proof is complete. □

With [8, Proposition 3.4] at hand, we immediately obtain the following

Theorem 3.3. *Let $\mathcal{A}$ be a unital C^*-algebra. For any faithful state ω of $\mathcal{A}$, $\sigma_2^\omega(a)$ is a strongly Leibniz seminorm.*

Alternatively, for any faithful tracial state ω, we can define a derivation on a Banach algebra to infer the above corollary. In fact, let us consider the Banach space

$$\mathcal{A}\oplus L^2(\mathcal{A},\omega)$$

endowed with the norm $\|(x,y)\| = \max(\|x\|,\|y\|_2)$. The linear operators

$$E\colon (x,y)\mapsto (0,\omega(x)\mathbf{1}_{\mathcal{A}})$$

and

$$T_a\colon (x,y)\mapsto (xa,ya)$$

on $\mathcal{A}\oplus L^2(\mathcal{A},\omega)$ define a strongly Leibniz seminorm L on $\mathcal{A}$ via the norm of the derivation $L(a)=\|\delta(T_a)\| = \|[E,T_a]\|$. From Lemma 3.1, we have that

$$\|\delta(T_a)(x,y)\| = \|(0,\omega(x)a-\omega(xa))\| \le \|a-\omega(a)\|_2\|x\| \le \|a-\omega(a)\|_2\|(x,y)\|.$$

With the choice of $(\mathbf{1}_{\mathcal{A}},0)$, we get

$$\|\delta(T_a)\| = \|a-\omega(a)\|_2.$$

Since ω is tracial, $\|xa\|_2 \le \|a\|\|x\|_2$. Hence it clearly follows that $\|T_a\| = \|a\|$. Notice that $T_{ab}=T_aT_b$. Now a direct application of the derivation rules gives that $\|\delta(T_a)\| = L(a) = \sigma_2^\omega(a)$ is a strongly Leibniz seminorm.

References

[1] K.M.R. Audenaert, Variance bounds, with an application to norm bounds for commutators, *Linear Algebra Appl.*, **432** (2010), 1126–1143.

[2] T. Bhattacharyya and P. Grover, Characterization of Birkhoff–James orthogonality, *J. Math. Anal. Appl.*, **407** (2013), 350–358.

[3] C. Franchetti, The norm of the minimal projection onto hyperplanes in $L_p[0,1]$ and the radial constant, *Boll. Un. Mat. Ital. B*, (7) 4, (1990), 803–821.

[4] Z. Léka, A note on central moments in C^*-algebras, http://arxiv.org/abs/1402.6528, to appear in J. Math. Inequal.

[5] A. W. Marshall, I. Olkin, B. C. Arnold, *Inequalities: Theory of Majorization and Its Applications*, second edition, Springer, 2011.

[6] S. Rolewicz, On minimal projections of the space $L_p[0,1]$ on 1-codimensional subspace, *Bull. Polish Acad. Sci. Math.*, **34** (1986), 151–153.

[7] M.A. Rieffel, Leibniz seminorms and best approximation from C^*-subalgebras, *Sci. China Math.*, **54** (2011), 2259–2274.

[8] M.A. Rieffel, Standard deviation is a strongly Leibniz seminorm, *New York J. Math.*, **20** (2014), 35–56.

Department of Applied Analysis, Eötvös Loránd University, H-1117 Budapest, Pázmány P. sétány 1/C, Hungary

E-mail address: badam@cs.elte.hu

Alfréd Rényi Institute of Mathematics, 1053 Budapest, Reáltanoda u. 13-15

E-mail address: leka.zoltan@renyi.mta.hu