# THE HASEGAWA-PETZ MEAN: PROPERTIES AND INEQUALITIES

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Abstract. We study a family of means introduced by H. Hasegawa and D. Petz (1996) [13]. Properties with respect to the parameter, such as monotonicity and logarithmic concavity, further, monotonicity and concavity in the mean variables are shown. Besides, the comparison between the Hasegawa-Petz mean and the geometric mean is completely solved. The connection to earlier results on operator monotonicity and some applications are also discussed.

## 1. Introduction

The theory of means has always been an extensive area of research. Means are closely connected to inequalities, therefore, means are basic to many applications to other fields. Probably the best-known means are the arithmetic, geometric and harmonic means:

$$A(x,y)=\frac{x+y}{2},\qquad G(x,y)=\sqrt{xy},\qquad H(x,y)=\frac{2xy}{x+y}.$$

However, there are numerous other means of interest. For instance, a more complicated family of means introduced by K. B. Stolarsky in [22], which now bears his name, is the following:

$$S_p(x,y)=\begin{cases}\left(\frac{x^p-y^p}{p(x-y)}\right)^{\frac{1}{p-1}}, & \text{if } p\neq 0, p\neq 1, x\neq y,\\ \frac{x-y}{\log x-\log y}, & \text{if } p=0, x\neq y,\\ \exp\left(\frac{x\log x-y\log y}{x-y}-1\right), & \text{if } p=1, x\neq y,\\ x, & \text{if } x=y.\end{cases}$$

We note that the means obtained as the limiting cases $p=1$ and $p=0$ are the so-called *logarithmic* and *identric mean*, respectively. In the past few decades, several properties of the Stolarsky mean, including monotonicity, Schur-convexity and comparison were investigated, see [22, 23, 18, 19]. For a comprehensive collection of means and their properties we refer to the monograph [9].

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In the papers [13, 14], H. Hasegawa and D. Petz introduced the following family of means for $-1 \le p \le 2$:

$$m_p(x,y) = \begin{cases} p(1-p)\frac{(x-y)^2}{(x^p-y^p)(x^{1-p}-y^{1-p})}, & \text{if } p \neq 0, p \neq 1, x \neq y, \\ \frac{x-y}{\log x - \log y}, & \text{if } p = 0 \text{ or } p = 1, x \neq y, \\ x, & \text{if } x = y. \end{cases} \quad (1.1)$$

We see that, for $p \neq 0$ and $p \neq 1$, $m_p$ is the weighted geometric mean of two Stolarsky means:

$$m_p(x,y) = S_p(x,y)^{1-p} S_{1-p}(x,y)^p. \quad (1.2)$$

Petz and Hasegawa were interested in applications to information theory and entropies (the mean $m_p$ is connected to the so-called Wigner-Yanase-Dyson information metric, see [13, 14, 27]), so they did not study any properties of $m_p$ as a mean. The only property which was investigated is the so-called *operator monotonicity*. Later, this property was shown in several other ways, see [10, 13, 14, 24, 25]. However, $m_p$ got few attention from the mean theoretic point of view, we are aware only the paper [2] which studies some properties of $m_p$ apart from operator monotonicity.

The aim of the present paper is the detailed analysis of the mean (1.1). We shall first recall the abstract notion of *mean* and some properties of abstract means. In Section 3, it will be proven that $m_p$ is a mean provided that $-1 \le p \le 2$. Then we show some properties, such as monotonicity and logarithmic concavity of $m_p$ with respect to the parameter $p$, further, monotonicity and concavity in the variables $x, y > 0$. Section 5 is devoted to some comparison results between $m_p$ and the arithmetic, geometric and harmonic means. Finally, miscellaneous topics related to the function (3.1) will be briefly discussed in Section 6. We recall the notion of operator monotonicity and compare the results regarding the operator monotonicity of the Hasegawa-Petz mean to ours. We also give an application to the comparison results of Section 5 by means of the so-called mean matrices. The presentation is self-contained and mostly elementary.

## 2. Means and their properties

A continuous function $m\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ (where $\mathbb{R}^+$ denotes the set of positive real numbers) is called a *mean* if

$$\min(x,y) \le m(x,y) \le \max(x,y) \quad (x, y \in \mathbb{R}^+). \quad (2.1)$$

In what follows, we shall refer to (2.1) as *internality property*. A mean is *symmetric* if $m(x,y) = m(y,x)$ for all $x, y > 0$, further, it is said to be *homogeneous* if $m(\lambda x, \lambda y) = \lambda m(x,y)$ for all $\lambda, x, y > 0$, and it is *strict* if both inequalities in (2.1) are strict for $x \neq y$. A homogeneous mean (actually, the internality property is not necessary) is uniquely determined by the function $f(x) = m(1,x)$ through $m(x,y) = xf(\frac{y}{x})$, and vice versa. Following the terminology of [17], we shall call $f$ the *representing function* of $m$. (In [7], $m(1,x)$ is called the *trace* of $m$.) By (2.1), we have $m(x,x) = x$ hence $f(1) = 1$. In addition, it is easily seen that a homogeneous mean is symmetric if and only if its representing function satisfies the equation

$$f\left(\frac{1}{x}\right) = \frac{1}{x} f(x) \quad (x > 0). \tag{2.2}$$

It is worthwhile to remark that the monotonicity and convexity properties of a symmetric homogeneous mean $m(x, y)$ in $x > 0$ for fixed $y > 0$, and in $y > 0$ for fixed $x > 0$, are equivalent to the monotonicity and convexity properties of its representing function. Further, the internality property is equivalent to

$$1 \leq f(x) \leq x \quad (x \geq 1). \tag{2.3}$$

Indeed, the symmetry property (2.2) implies that

$$x \leq f(x) \leq 1 \quad (0 < x \leq 1)$$

thus by the homogeneity (2.1) follows. Finally, we mention that obviously the arithmetic, geometric and harmonic means are indeed strict means, and their representing functions are

$$f_A(x) = \frac{x+1}{2}, \qquad f_G(x) = \sqrt{x}, \qquad f_H(x) = \frac{2x}{x+1}.$$

For further details on means and their properties, see [7, 9].

## 3. Internality property

We first notice that $m_p(x, y)$ is a symmetric homogeneous function of $x, y > 0$ so it is uniquely determined by the following function:

$$f_p(x) = \begin{cases} p(1-p)\frac{(x-1)^2}{(x^p-1)(x^{1-p}-1)}, & \text{if } p \neq 0, p \neq 1, x \neq 1, \\ \frac{x-1}{\log x}, & \text{if } p = 0 \text{ or } p = 1, x \neq 1, \\ 1, & \text{if } x = 1. \end{cases} \tag{3.1}$$

Note that $m_p(x, y)$ is symmetric in $p$ with respect to the point $1/2$, that is, $m_p(x, y) = m_{1-p}(x, y)$ for all $x, y > 0$, therefore, we may always suppose $p \geq 1/2$. For some special cases of the parameter we obtain some well-known means, for $p = 0$ and $p = 1$ the logarithmic mean, for $p = -1$ and $p = 2$ the harmonic mean and

$$m_{1/2}(x, y) = \left(\frac{\sqrt{x} + \sqrt{y}}{2}\right)^2$$

which is called the *power mean* with exponent $1/2$ (which could easily seen to be a strict mean).

Our first task is to answer the question: for which $p$ is $m_p$ a mean? For $0 < p < 1$ we can use the identity (1.2). Since $S_\alpha$ is a mean, for $0 < \alpha < 1$ we have

$$(\min(x, y))^{1-\alpha} \leq S_\alpha(x, y)^{1-\alpha} \leq (\max(x, y))^{1-\alpha},$$

therefore, by taking the product of $S_p(x, y)^p$ and $S_{1-p}(x, y)^{1-p}$, it follows that $m_p$ also has the internality property for $0 < p < 1$. Passing to the limit implies that for $p = 0$ and $p = 1$ it is also a mean. We recall from [22] that the internality property for the Stolarsky mean, for $\alpha \neq 0$ and $\alpha \neq 1$, follows from the mean value theorem for the monotone increasing function $g(x) = x^\alpha$:

$$\left(\frac{x^\alpha - y^\alpha}{\alpha(x-y)}\right)^{\frac{1}{\alpha-1}} = (g')^{-1}\left(\frac{g(x)-g(y)}{x-y}\right) = (g')^{-1}(g'(\xi)) = \xi$$

where $\min(x,y) < \xi < \max(x,y)$. If $\alpha = 0$, one should take the function $g(x) = \log x$. Note that the above arguments also show that $S_p$ and $m_p$ are strict means. (In view of the above property, the Stolarsky mean is a so-called Lagrangian mean.)

Although, the above arguments implied that $m_p$ is a mean for $0 \le p \le 1$, it does not tell anything for the other cases. However, observe that $m_p$ is not a mean for $p > 2$ and symmetrically for $p < -1$. Indeed, for fixed $y > 0$ and $p > 2$,

$$\lim_{x\to+\infty} m_p(x,y) = \lim_{x\to+\infty} p(p-1)\frac{x^{p+1}}{x^{2p-1}} = \lim_{x\to+\infty} p(p-1)x^{2-p} = 0, \quad (3.2)$$

thus $m_p$ violates the internality property. This means that $m_p$ could be a mean only for $-1 \le p \le 2$. The affirmative answer is given below.

**Proposition 1.** *The internality property is satisfied by $m_p$ if and only if $-1 \le p \le 2$.*

*Proof.* Suppose $p \ne 0$ and $p \ne 1$. It suffices to show (2.3). To this end, it is convenient to substitute $x = \exp(2\lambda)$ where $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} f_p(e^{2\lambda}) &= \frac{p(1-p)\left(e^\lambda \frac{e^\lambda - e^{-\lambda}}{2}\right)^2}{e^{\lambda p}\frac{e^{\lambda p}-e^{-\lambda p}}{2} e^{\lambda(1-p)}\frac{e^{\lambda(1-p)}-e^{-\lambda(1-p)}}{2}} \\ &= \frac{p(1-p)e^\lambda \sinh^2(\lambda)}{\sinh(\lambda p)\sinh(\lambda(1-p))}, \end{aligned} \quad (3.3)$$

hence (2.3) reduces to the following inequality:

$$e^{-\lambda} \le \frac{p(1-p)\sinh^2(\lambda)}{\sinh(\lambda p)\sinh(\lambda(1-p))} \le e^\lambda \quad (\lambda \ge 0). \quad (3.4)$$

Note that for $\lambda \le 0$ reverse inequalities must hold in (3.4). Since there is equality in (3.4) for $\lambda = 0$, it suffices to show that the same inequality holds for the logarithmic derivative of each sides, that is,

$$-1 \le 2\coth(\lambda) - p\coth(p\lambda) - (1-p)\coth((1-p)\lambda) \le 1 \quad (\lambda > 0). \quad (3.5)$$

(Notice that (3.5) should also hold for $\lambda < 0$.) We claim that the function

$$g(p) := p\coth(p\lambda) + (1-p)\coth((1-p)\lambda) \quad (\lambda > 0)$$

is monotone decreasing for $p \le 0$, increasing for $p \ge 0$ and it is convex. It is well knwon that the function $x \mapsto x\coth(x)$ is infinitely differentiable (since $x/\sinh(x)$ is as) so $g$ is also infinitely differentiable also at $p = 0$ and $p = 1$. On the other hand, due to the well-known inequality $\tanh(y) < y$ ($y > 0$),

$$\begin{aligned} g''(p) &= \frac{2\lambda}{\sinh^3(p\lambda)}(p\lambda - \tanh(p\lambda)) \\ &\quad + \frac{2\lambda}{\sinh^3((1-p)\lambda)}((1-p)\lambda - \tanh((1-p)\lambda)) > 0 \quad (\lambda > 0), \end{aligned}$$

whence the convexity of $g$ follows. Further, $g(p) = g(1-p)$ thus $g'(1/2) = 0$ and therefore, because of the convexity of $g$, $g'(p) \geq 0$ for $p \geq 0$ and $g'(p) \leq 0$ for $p \leq 0$. Now, due to the monotonicity of the function $g$, it suffices to verify (3.5) for $p = 1/2$ and $p = 2$. So it remains to show that

$$2\coth(\lambda) - \coth\left(\frac{1}{2}\lambda\right) \leq 1 \quad (\lambda > 0)$$

and

$$-1 \leq \coth(\lambda) - 2\coth(2\lambda) \quad (\lambda > 0) \tag{3.6}$$

which are obviously equivalent. Since

$$2\coth(2\lambda) = \coth(\lambda) + \tanh(\lambda)$$

thus

$$\coth(\lambda) - 2\coth(2\lambda) = -\tanh(\lambda) \geq -1$$

which is exactly (3.6). The proof is complete. We remark that the above arguments also imply that $m_p$ is a strict mean for $-1 \leq p \leq 2$. □

*Remark* 2. In Section 4, we shall obtain another proof for the internality property by using the monotonicity of $m_p$ with respect to $p$. However, it might be desirable to have a direct proof.

In the sequel, according to Proposition 1, we shall refer to $m_p$ for $-1 \leq p \leq 2$ as the Hasegawa-Petz mean with parameter $p$.

## 4. Monotonicity, concavity, log-concavity

In this section, we shall show some properties of the Hasegawa-Petz mean. We begin with the behavior with respect to the parameter $p$.

**Proposition 3.** *For fixed $x, y > 0$, the function $p \mapsto m_p(x,y)$ is increasing in $[1/2, +\infty)$ and decreasing in $(-\infty, 1/2]$, further, it is logarithmically concave in $\mathbb{R}$ (i.e., $p \mapsto \log m_p(x,y)$ is a concave function).*

*Proof.* We show that $f_p(x)$ has the desired properties. As in the proof of Proposition 1, we substitute $x = \exp(2\lambda)$ where $\lambda \in \mathbb{R}$. Then, with the help of (3.3), and of the addition formula

$$\sinh(a)\sinh(b) = \frac{1}{2}\cosh(a+b) - \frac{1}{2}\cosh(a-b), \tag{4.1}$$

we find that

$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{1}{f_p(e^{2\lambda})}\right) = \frac{\lambda p(1-p)\sinh(\lambda(1-2p)) - \sinh(\lambda p)\sinh(\lambda(1-p))(1-2p)}{(p(1-p))^2 e^{\lambda}\sinh^2(\lambda)}.$$

Thus it suffices to prove that

$$\lambda p(1-p)\sinh(\lambda(1-2p)) - \sinh(\lambda p)\sinh(\lambda(1-p))(1-2p) \geq 0. \tag{4.2}$$

If $1/2 \leq p \leq 1$, then (4.2) is equivalent to

$$\frac{\sinh(\lambda p)\sinh(\lambda(1-p))}{p(1-p)} \geq \frac{\lambda\sinh(\lambda(1-2p))}{1-2p}. \tag{4.3}$$

By Cauchy's mean value theorem (applied in the variable $p$) and by (4.1),

$$\frac{\sinh(\lambda p)\sinh(\lambda(1-p))}{p(1-p)} = \frac{\lambda\sinh(\lambda(1-2\xi))}{1-2\xi}$$

for some $\xi \in (p,1)$. Since the function $g(y) = \lambda\sinh(\lambda y)$ is concave for $y \le 0$ and for fixed $\lambda \in \mathbb{R}$, further, $g(0) = 0$ thus the divided difference function of $g$ centered at the origin, i.e., $y \mapsto \lambda\sinh(\lambda y)/y$ is an increasing function for $y \le 0$. Hence, due to $\xi \in (p,1)$ and $p \ge \frac{1}{2}$,

$$\frac{\lambda\sinh(\lambda(1-2\xi))}{1-2\xi} \ge \frac{\lambda\sinh(\lambda(1-2p))}{1-2p}, \tag{4.4}$$

which yields (4.3). If $p \ge 1$ then in (4.3) we have reversed inequality, further, $\xi \in (1,p)$ so (4.4) is also reversed. The case $p \le \frac{1}{2}$ follows by symmetry.

Let us turn to the log-concavity. By an easy calculation we obtain that

$$\frac{d^2}{dp^2}\log\left(\frac{p\sinh(\lambda)}{\sinh(p\lambda)}\right) = \frac{(p\lambda)^2 - \sinh^2(p\lambda)}{p^2\sinh^2(p\lambda)} \le 0 \quad (p,\lambda \in \mathbb{R}), \tag{4.5}$$

due to the well-known inequality $\sinh^2(y) \ge y^2$ $(y \in \mathbb{R})$. Now the logarithmic concavity of $f_p(e^{2\lambda})$ and so of $f_p(x)$ follows. □

**Corollary 4.** *The two-variable function $m_p$ defined by* (1.1) *is a mean if* $-1 \le p \le 2$.

*Proof.* By Proposition 3,

$$\min(x,y) \le H(x,y) = m_2(x,y) \le m_p(x,y) \le m_{1/2}(x,y) \le \max(x,y)$$

so that the internality property holds true. □

After having recovered the behavior of $m_p(x,y)$ with respect to the parameter $p$, it is worthwhile to show some properties also in the variables $x, y > 0$.

**Proposition 5.** *For fixed $-1 \le p \le 2$, the function $m_p(x,y)$ is monotone increasing and concave in both variables $x, y > 0$.*

*Proof.* It suffices to prove that $f_p(x)$ is monotone increasing and concave in $x > 0$ for fixed $-1 \le p \le 2$. To show the monotonicity, we use the idea of [10]. Observe that if $p \ne 0$ and $p \ne 1$, then

$$f_p(x) = p(p-1)\frac{x-1}{g_p(x)-1}$$

where

$$g_p(x) = \begin{cases} \frac{x^p-1}{x-1} + \frac{x^{1-p}-1}{x-1}, & \text{if } p \ne 0, p \ne 1, x \ne 1, \\ 1, & \text{if } x = 1. \end{cases}$$

Since

$$\frac{x^\alpha - 1}{x-1} = \alpha\int_0^1 (tx+1-t)^{\alpha-1}dt \tag{4.6}$$

and $(tx+1-t)^{\alpha-1}$ is a convex function of $x > 0$ for fixed $0 \le t \le 1$ and $0 < \alpha < 1$, therefore, $(x^\alpha - 1)/(x-1)$ is also a convex function of $x > 0$ for $0 < \alpha < 1$ thus $g_p(x)$ is convex in $x > 0$ for $0 < p < 1$. The divided difference function of a convex function is monotone increasing hence $(g_p(x)-1)/(x-1)$ is increasing whence it follows that $f_p(x)$ is also increasing in $x > 0$ for

$0 < p < 1$. If $1 < p \le 2$, then $g_p(x)$ is concave in $x > 0$ and so is $f_p(x)$. The cases $p = 1$ and $p = 0$ follow by passing to the limit.

Now let us turn to the concavity. We first consider the case $0 < p < 1$. Denoting $F_p(x) = 1/f_p(x)$, we have

$$f_p''(x) = -\frac{F_p''(x)F_p(x) - 2(F_p'(x))^2}{F_p^2(x)}$$

thus it suffices to show that

$$2(F_p'(x))^2 \le F_p''(x)F_p(x). \tag{4.7}$$

In view of the integral representation (4.6), we have

$$F_p(x) = \int_0^1 \int_0^1 (tx + 1 - t)^{p-1}(sx + 1 - s)^{-p} dt\, ds.$$

For simplicity, we denote

$$h(x) := (tx + 1 - t)^{p-1}, \qquad k(x) := (sx + 1 - s)^{-p}.$$

After differentiation under the integral sign, inequality (4.7) reduces to

$$\left(\int_0^1 \int_0^1 \sqrt{2}(hk)'\right)^2 \le \int_0^1 \int_0^1 (hk)'' \cdot \int_0^1 \int_0^1 hk. \tag{4.8}$$

Notice that for $0 < p < 1$, $h$ and $k$ are both convex and monotone decreasing positive functions thus $hk$ is also convex and decreasing hence $(hk)' \le 0$ and $(hk)'' \ge 0$. If we show that

$$\sqrt{2}|(hk)'| \le \sqrt{(hk)''}\sqrt{hk}, \tag{4.9}$$

then (4.8) follows by the Cauchy-Schwarz inequality. After some simple calculation we obtain that inequality (4.9) is equivalent to

$$2h'hk'k \le \left(h''h - 2(h')^2\right) k^2 + \left(k''k - 2(k')^2\right) h^2. \tag{4.10}$$

We verify that

$$\left(h''h - 2(h')^2\right)\left(k''k - 2(k')^2\right) = (h')^2(k')^2, \tag{4.11}$$

then the inequality of the arithmetic and geometric means implies (4.10). The identity (4.11) is just an easy calculation:

$$(h')^2 = (p-1)^2(tx + 1 - t)^{2p-4}, \quad h''h - 2(h')^2 = p(1-p)(tx + 1 - t)^{2p-4},$$

and

$$(k')^2 = p^2(sx + 1 - s)^{-2p-2}, \quad k''k - 2(k')^2 = p(1-p)(sx + 1 - s)^{-2p-2},$$

which immediately yield (4.10).

Now let $1 < p \le 2$. Then

$$f_p(x) = p(p-1)\frac{x^{p-1}(x-1)^2}{(x^p - 1)(x^{p-1} - 1)} = p(p-1)\frac{x-1}{x^p - 1} - p(p-1)\frac{x-1}{x^p - 1 - 1}.$$

As the integral representation (4.6) shows, the function $(x^{p-1} - 1)/(x - 1)$ is convex, further, $(x^p - 1)/(x - 1)$ is concave for $1 < p \le 2$, therefore, $f_p$ is also concave. The cases $p = 0$ and $p = 1$ follow by passing to the limit. This completes the proof. $\square$

*Remark* 6. Let us remark that for $p > 2$ (and symmetrically for $p < -1$) the function $f_p$ is neither monotone nor concave, since (3.2) shows that in these cases $0 < f_p(x) \to 0$ as $x \to +\infty$.

## 5. The Hasegawa-Petz mean and the geometric mean

In what follows, we are interested in the comparison of the Hasegawa-Petz mean and the arithmetic, geometric and harmonic means. The results may be interesting in itself, however, they have applications which will be discussed in Section 6.

We begin with the easy part. Since $m_p$ is decreasing in $p \geq 1/2$ and $m_2(x,y) = H(x,y)$, therefore,

$$m_p(x,y) \geq H(x,y) \quad (-1 \leq p \leq 2)$$

where the inequality is strict whenever $x \neq y$ and $p \neq 2$, $p \neq -1$. On the other hand, $m_{1/2}(x,y) \leq A(x,y)$ which is an elementary exercise to show, thus

$$m_p(x,y) \leq A(x,y) \quad (-1 \leq p)$$

with strict inequalities if $x \neq y$.

Now, let us turn to the harder part, the Hasegawa-Petz mean and the geometric mean. It is well known (see, e.g., [8]) that the logarithmic mean is larger than or equal to the geometric mean. Since $m_1(x,y) = L(x,y)$ and $m_{1/2}(x,y) \geq G(x,y)$, which is again an easy exercise to prove, thus

$$m_p(x,y) \geq G(x,y) \quad (0 \leq p \leq 1) \tag{5.1}$$

with strict inequalities whenever $x \neq y$. We note that (5.1) was proved in [2] by applying the inequality

$$p\frac{x-1}{x^p-1} \geq x^{\frac{1-p}{2}} \quad (x > 0)$$

which is equivalent to $S_p(x,y) \geq G(x,y)$ and can be verified directly (as in [2]) or using the monotonicity properties of the Stolarsky mean in $p$ (similarly as for $m_p$ above), see [22]. Clearly, $m_2 = H(x,y) \leq G(x,y)$ so the question is: what is the largest $p$ such that (5.1) holds? Besides, what is the smallest $p$ for which reverse inequality holds in (5.1) and what happens in between?

To answer these questions it suffices to study the comparison between the representing functions $f_p(x)$ and $f_G(x) = \sqrt{x}$. First, it is worthwhile to consider the ratio $f_p(x)/\sqrt{x}$ as $x \to +\infty$. Obviously, $f_p(x)$ is of order $x^{2-p}$ for $p > 1$ and for large $x > 0$ hence the ratio $f_p(x)/\sqrt{x}$ tends to 0 as $x \to +\infty$ for $p > 3/2$. This means that for $p > 3/2$ the inequality (5.1) cannot be true, either the reverse holds, or the two means are not comparable.

In order to gain more insight into the relation of the two means we use Taylor expansion of the ratio $f_p(x)/\sqrt{x}$. As we did earlier, it is convenient to substitute $x = \exp(2\lambda)$ ($\lambda \in \mathbb{R}$). Then

$$\frac{f_p(e^\lambda)}{e^\lambda} = \frac{p(1-p)\sinh^2(\lambda)}{\sinh(\lambda p)\sinh(\lambda(1-p))}.$$

By using the expansion

$$\sinh(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^{2n+1}}{(2n+1)!} \tag{5.2}$$

it follows that

$$\sinh^2(\lambda) = \lambda^2 + \frac{1}{3}\lambda^4 + o(\lambda^4)$$

and

$$\frac{\sinh(p\lambda)\sinh((1-p)\lambda)}{p(1-p)} = \lambda^2 + \frac{1}{6}\left(p^2 + (1-p)^2\right)\lambda^4 + o(\lambda^4).$$

Therefore, for small $\lambda$, $f_p(e^\lambda)/e^\lambda \geq 1$ provided that $p^2 + (1-p)^2 \leq 2$, that is, $1/2 - \sqrt{3}/2 \leq p \leq 1/2 + \sqrt{3}/2$, otherwise reverse inequality holds. So inequality (5.1) could hold only for $1/2 - \sqrt{3}/2 \leq p \leq 1/2 + \sqrt{3}/2$. Now we are ready to state the main result of this section.

**Theorem 7.** *The following inequalities hold true:*

$$\begin{aligned} m_p(x,y) &\geq G(x,y) \qquad (1/2 - \sqrt{3}/2 \leq p \leq 1/2 + \sqrt{3}/2), \\ m_p(x,y) &\leq G(x,y) \qquad (3/2 \leq p, \ \text{or}\ p \leq -1/2) \end{aligned}$$

*with strict inequalities whenever $x \neq y$. For $1/2 + \sqrt{3} < p < 3/2$ (and symmetrically for $-1/2 < p < 1/2 - \sqrt{3}$) the two means are not comparable.*

*Proof.* The last statement follows from the arguments which precede the theorem, for $1/2 + \sqrt{3} < p < 3/2$ and for small $x > 0$ we have $f_p(x) > \sqrt{x}$, and for large $x$ the reverse holds.

Let us consider first the case $3/2 \leq p$. We show that $f_p(e^{2\lambda}) \leq e^\lambda$ or equivalently (for $p \neq 1$)

$$\frac{\sinh(p\lambda)}{p} \cdot \frac{\sinh((1-p)\lambda)}{1-p} \geq \sinh^2(\lambda). \tag{5.3}$$

Analogously to (4.5), it is easily seen that the function $p \mapsto p/\sinh(p\lambda)$ is logarithmically concave for fixed $\lambda > 0$. Therefore, the function

$$p \mapsto \frac{\sinh(p\lambda)}{p}$$

is logarithmically convex for $\lambda > 0$. Further, it is also monotone increasing for $\lambda > 0$, since

$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{\sinh(p\lambda)}{p}\right) = \frac{p\lambda\cosh(p\lambda) - \sinh(p\lambda)}{p^2} \geq 0,$$

due to the well-known inequality $\tanh(y) \leq y$ for $y \geq 0$. The logarithmic convexity and the monotonicity imply for $p \geq 3/2$ that

$$\begin{aligned} \log\left(\frac{\sinh(p\lambda)}{p}\right) + \log\left(\frac{\sinh((p-1)\lambda)}{p-1}\right) &\geq 2\log\left(\frac{\sinh((p-\frac{1}{2})\lambda)}{p-\frac{1}{2}}\right) \\ &\geq 2\log\sinh(\lambda) \end{aligned}$$

which is, after rearrangement, equivalent to (5.3) hence also to $f_p(e^{2\lambda}) \leq e^\lambda$. Since $f_p(e^{2\lambda})/e^\lambda$ is an even function in $\lambda$, $f_p(e^{2\lambda}) \leq e^\lambda$ holds for every $\lambda \in \mathbb{R}$ (and by passing to the limit also for $p = 0$).

Now, let $1/2 - \sqrt{3}/2 \leq p \leq 1/2 + \sqrt{3}/2$. It suffices to show that $f_p(e^{2\lambda}) \geq e^\lambda$, moreover, due to the monotonicity (and symmetry) of $f_p(e^{2\lambda})$

with respect to $p$ (see Proposition 3), we may restrict the proof to the case $p = 1/2 + \sqrt{3}/2$. Then $f_p(e^{2\lambda}) \geq e^\lambda$ reduces to

$$g(\lambda) := \frac{1}{2}\sinh^2(\lambda) + \sinh\left(\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)\lambda\right)\sinh\left(\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right)\lambda\right) \geq 0. \quad (5.4)$$

Since in (5.4) equality holds for $\lambda = 0$ so the inequality will follow by showing that $g'(\lambda) \geq 0$ for $\lambda > 0$ and $g'(\lambda) \leq 0$ for $\lambda < 0$. By an easy calculation, with the help of (4.1) and (5.2), we obtain that

$$\begin{aligned} g'(\lambda) &= \frac{1}{2}\sinh(2\lambda) + \frac{1}{2}\sinh(\lambda) - \frac{\sqrt{3}}{2}\sinh(\sqrt{3}\lambda) \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left(2^{2n+1} + 1 - 3^{n+1}\right)\frac{\lambda^{2n+1}}{(2n+1)!}. \end{aligned}$$

We verify that

$$c_n := 2^{2n+1} + 1 - 3^{n+1} \geq 0 \quad (n = 0, 1, \dots),$$

then $g'(\lambda) \geq 0$ follows for $\lambda > 0$ and $g'(\lambda) \leq 0$ for $\lambda < 0$. We have $c_0 = 0$ and $c_1 = 0$, further, the monotonicity of the sequence $(4/3)^n$ implies for $n \geq 2$ that $2 \cdot 2^{2n} > 3 \cdot 3^n$, so that $c_n > 0$. This completes the proof. □

## 6. Miscellanea

We discuss some applications and some further properties of the function $f_p$ given by (3.1). The results are not new, however, it is worthwhile to present them due to the connection to our results.

6.1. **Integral representation.** Let us first mention an integral representation of $1/f_p$ established by Hasegawa and Petz in [13]. The following result from functional analysis (see [3, 26])

$$z^{\beta-1} = \frac{\sin\beta\pi}{\pi}\int_0^\infty \frac{\lambda^{\beta-1}}{\lambda + z}d\lambda \quad (0 < \beta < 1)$$

combined with (4.6) yields for $0 < p < 1$ that

$$\frac{1}{f_p(x)} = \frac{\sin p\pi}{\pi}\int_0^\infty \lambda^{p-1}\int_0^1\int_0^1 \frac{1}{x((1-t)\lambda + 1 - s) + (t\lambda + s)}dt\,ds\,d\lambda.$$

From this integral representation the monotonicity of $f_p(x)$ with respect to $x > 0$ easily follows (for $0 < p < 1$).

6.2. **Operator monotonicity.** The most studied property of $f_p$ in the literature is the so-called operator monotonicity, see [10, 13, 14, 24, 25]. Operator monotone functions were introduced by K. Löwner in 1934 in the seminal paper [20]. They have a wide spectrum of applications, for instance, in quantum information theory [21], or in the topic of operator means [17]. For its importance, we recall the definition of operator monotonicity, see [3] for details.

Let $\mathbb{H}_n$ denote the space of $n \times n$ complex Hermitian matrices with the usual ordering: $A \leq B$ means that $B - A$ is positive-semidefinite. Let $f$ be a real function on an interval $I$. If $D$ is a diagonal matrix with diagonal entries $\lambda_j \in I$ $(j = 1, \dots, n)$, then $f(D) := \operatorname{diag}(f(\lambda_1), \dots, f(\lambda_n))$. If $A$ is a

Hermitian matrix with eigenvalues in $I$, then let $f(A) := Uf(D)U^*$ where $A = UDU^*$ with unitary matrix $U$ and diagonal matrix $D$ containing the eigenvalues of $A$.

**Definition 1.** A function $f: I \to \mathbb{R}$ is called *matrix monotone of order $n$* if $A \le B$ implies $f(A) \le f(B)$ for every $A, B \in \mathbb{H}_n$ having eigenvalues in the interval $I$. If $f$ is matrix monotone of order $n$ for all $n$, then $f$ is called *operator monotone* (on $I$).

The main result in the literature is the following.

**Theorem 8.** *The function $f_p: \mathbb{R}^+ \to \mathbb{R}^+$ defined by* (3.1) *is operator monotone if and only if $-1 \le p \le 2$.*

Obviously, the operator monotonicity implies the monotonicity of $f_p$. Moreover, a famous result of F. Hansen and G. K. Pedersen [12] says that a function $\mathbb{R}^+ \to \mathbb{R}^+$ is operator monotone if and only if it is operator concave, that is,

$$f(\lambda A + (1-\lambda)B) \ge \lambda f(A) + (1-\lambda) f(B)$$

for every $0 \le \lambda \le 1$, $A, B \in \mathbb{H}_n$ having positive eigenvalues ($n \in \mathbb{N}$ being arbitrary). Whence it follows that $f_p$ is also concave in $x > 0$, which coincides with our results.

As an application of the comparison with the arithmetic and harmonic means, we recall the following result of [17].

**Theorem 9.** *If $f: \mathbb{R}^+ \to \mathbb{R}^+$ is an operator monotone function such that $f(1) = 1$ and $f(1/x) = f(x)/x$ then*

$$\frac{2x}{x+1} \le f(x) \le \frac{x+1}{2}. \tag{6.1}$$

Clearly, inequality (6.1) implies (2.3), therefore, an operator monotone function $f: \mathbb{R}^+ \to \mathbb{R}^+$ with the properties $f(1) = 1$ and $f(1/x) = f(x)/x$ generates a symmetric homogeneous mean (moreover, an operator mean [17]). Theorem 9 says that this mean should lie between the arithmetic and harmonic means. Therefore, the function $f_p$ could be matrix monotone only if $-1 \le p \le 2$ so the "only if" part of Theorem 8 follows.

We close the results connected to operator monotonicity by mentioning an open problem of [1].

**Problem.** What can be said about the operator monotonicity of $(-f_p')$?

The operator monotonicity of $(-f_p')$ would imply the so-called *strong subadditivity* of $f_p$ which is related to strong subadditivity of entropy functions, see [1]. In [11], the strong subadditivity is called *submodularity.* We note that according to [1], numerical computations show that $(-f_p')$ is operator monotone for $-1 \le p \le 1$.

6.3. **Mean matrices.** Finally, let us see some applications of the comparison results of Section 5. Let $\lambda_1 < \cdots < \lambda_n$ be positive numbers and $m(x, y)$ be a symmetric mean. Consider the matrix $M$ with entries $m_{ij} = m(\lambda_i, \lambda_j)$ $(i, j = 1, \ldots, n)$. We may call $M$ a *mean matrix.* The positive-semidefiniteness of the matrix $M$ for all choices of $\lambda_i$ and for every $n \in \mathbb{N}$ is an interesting and often complicated problem. A more general question is the following: is the matrix with entries $m_{ij}^r$ positive definite for every $r > 0$ and for all choices of $\lambda_i$ and every $n \in \mathbb{N}$? This is the so-called *infinite divisibility* property of $M$. These problems are connected to operator means and comparison of their norms, see [4, 15, 16, 6]. For some examples of mean matrices we refer to [5, 2].

By taking $n = 2$,

$$M = \begin{bmatrix} \lambda_1 & m(\lambda_1, \lambda_2) \\ m(\lambda_1, \lambda_2) & \lambda_2 \end{bmatrix},$$

therefore, the positive-semidefiniteness of $M$, for $n = 2$, is equivalent to $\det M = \lambda_1\lambda_2 - m(\lambda_1, \lambda_2)^2 \geq 0$ which means that $m$ must be smaller than or equal to the geometric mean. However, this fact does not guarantee positive-semidefiniteness of $M$ for every $n \in \mathbb{N}$, see the cited papers. Likewise, instead of $M$ we may consider the matrix $N$ with entries $1/m(\lambda_i, \lambda_j)$. A necessary condition of the positive-semidefiniteness of the matrix $N$ (of all order) is that $m$ is larger than or equal to the geometric mean. By Theorem 7, it follows that the mean matrices $N_{f_p}$ could be positive-semidefinite only for $1/2 - \sqrt{3}/2 \leq p \leq 1/2 + \sqrt{3}$. The positive-semidefiniteness is verified for $0 \leq p \leq 1$ in [2]. The infinite divisibility of $N_{f_p}$ for $0 \leq p \leq 1$ follows from the infinite divisibility properties of the Stolarsky means shown in [5] and from the identity (1.2).

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