THE BROCARD ANGLE AND A GEOMETRICAL GEM FROM DMITRIEV AND DYNKIN

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Abstract. In a celebrated paper on the eigenvalues of special matrices, N. Dmitriev and E. Dynkin formulated and proved a nice geometrical lemma. This lesser-known gem provides a simple proof of the maximal value of the Brocard angle and yields also the solution to some challenging problems from the past few decades.

The Brocard points and Brocard angle of a triangle have attracted great attention since their discovery in the beginning of the 19th century. Many properties and their generalizations to polygons of these geometrical objects appear from time to time in various forms such as mathematical olympiad problems or challenging problems in expository mathematical journals. One frequently occuring question is the maximal value of the Brocard angle. In this note, we make an attempt to collect some of this problem's appearances and to link them to a hidden geometrical lemma from a 1945 paper on the location of eigenvalues of special matrices in the complex plane.

The Brocard Points and Angles

The history of the Brocard points began in 1816 when August Leopold Crelle (1780–1855) published a small book on plane triangles. He was dealing with a certain point P inside a triangle ABC such that the angles $\sphericalangle PAB$, $\sphericalangle PBC$, $\sphericalangle PCA$ are equal, see Fig. 1. Among other things, he showed that (for a given oder of vertices) there uniquely exists such a point P and the angle $\chi = \sphericalangle PAB = \sphericalangle PBC = \sphericalangle PCA$ satisfies

$$\chi = \cot \sphericalangle CAB + \cot \sphericalangle ABC + \cot \sphericalangle BCA. \tag{1}$$

By reversing the order of vertices we obtain another point P' with the same angle χ, see Fig. 1. These remarkable points of a triangle were studied some years after Crelle by Carl Friedrich Andreas Jacobi (1801–1875) who should not to be confused with the perhaps much famous Carl Gustav Jakob Jacobi (as some might do, by references not mentioning the first names of Jacobi). However, the points were named after the French geometer and military officer Henri Brocard (1845–1922) who rediscovered them in 1875 and drew again considerable attention to explore their many fascinating properties (see [7, 8] for a detailed exposition on this topic). Such a property, which is usually derived from the formula (1), is that the Brocard angle χ has a maximum value,

$$\chi \leq \frac{\pi}{6}, \tag{2}$$

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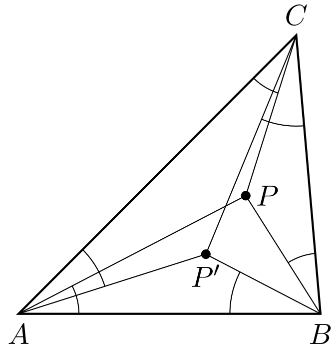

Figure 1. Brocard points

and equality can hold if only if the triangle is equilateral.

It seems lesser-known that a generalization of the inequality (2) was formulated in the paper [3] from 1945 by the Russian mathematician and physicist Nikolai Aleksandrovich Dmitriev (1924–2000) and mathematician Eugene Borisovich Dynkin (1924–). They stated the following nice geometrical result as a lemma in their paper (we keep the term lemma).

Lemma. *Let P be an arbitrary point in the interior of a convex n-gon $A_1A_2\dots A_n$ and denote $A_{n+1} = A_1$ (see Fig. 2). Then*

$$\min_{k=1,\dots,n} PA_kA_{k+1} \le \frac{\pi}{2} - \frac{\pi}{n}.$$

Equality occurs if and only if $A_1A_2\dots A_n$ is a regular n-gon.

In fact, the case of equality was not explicitly mentioned by Dmitriev and Dynkin, but it follows from their argument. We first present their proof by giving some details on a minor step that they left to the reader. Then we recall some related problems from the past few decades and solve them with the lemma of Dmitriev and Dynkin.

Proof of the Lemma

We prove the lemma by contradiction. Assume that $\sphericalangle PA_kA_{k+1} > \alpha$ for every $k = 1,\dots,n$ where we use the notation $\alpha = \pi/2 - \pi/n$ for brevity.

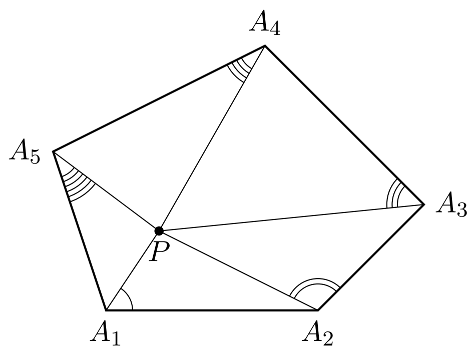

Figure 2. Minimal angle lemma

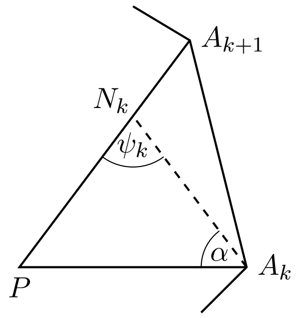

FIGURE 3. Proof of the Lemma

Since $\sphericalangle PA_kA_{k+1} > \alpha$, there is a point N_k on the segment PA_{k+1} such that $\sphericalangle PA_kN_k = \alpha$, see Fig. 3. Denote by ψ_k the angle $\sphericalangle PN_kA_k$. Then the sine theorem in the triangle PA_kN_k implies

$$\frac{\sin\psi_k}{\sin\alpha} = \frac{PA_k}{PN_k} > \frac{PA_k}{PA_{k+1}}.$$

The product of the above inequality for $k = 1, \ldots, n$ yields

$$\frac{\sin\psi_1}{\sin\alpha} \cdot \frac{\sin\psi_2}{\sin\alpha} \cdot \ldots \cdot \frac{\sin\psi_n}{\sin\alpha} > \frac{PA_1}{PA_2} \cdot \frac{PA_2}{PA_3} \cdot \ldots \cdot \frac{PA_n}{PA_1} = 1,$$

therefore,

$$\sin\psi_1 \cdot \ldots \cdot \sin\psi_n > (\sin\alpha)^n. \tag{3}$$

On the other hand

$$\begin{aligned}\sum_{k=1}^{n} \psi_k &= \sum_{k=1}^{n} (\pi - \alpha - \sphericalangle A_kPA_{k+1}) \\ &= n(\pi - \alpha) - \sum_{k=1}^{n} \sphericalangle A_kPA_{k+1} = n(\pi - \alpha) - 2\pi = n\alpha.\end{aligned}$$

But the product $\sin\psi_1 \cdot \ldots \cdot \sin\psi_n$ in which the angles ψ_k satisfy

$$0 < \psi_1 < \pi, \ldots, 0 < \psi_n < \pi, \quad \sum_{k=1}^{n} \psi_k = n\alpha,$$

attains its maximum for $\psi_1 = \cdots = \psi_n = \alpha$. So

$$\sin\psi_1 \cdot \ldots \cdot \sin\psi_n \le (\sin\alpha)^n$$

which contradicts (3). The value and location of the maximum of $\sin\psi_1 \cdot \ldots \cdot \sin\psi_n$ (which was not justified by Dmitriev and Dynkin) follows easily from the inequality of arithmetic and geometric means, combined with Jensen's inequality for the concave sine function on the interval $[0, \pi]$ (as they might also had the same reasoning in their minds). Indeed,

$$\begin{aligned}\sin\psi_1 \cdot \ldots \cdot \sin\psi_n &\le \left(\frac{\sin\psi_1 + \cdots + \sin\psi_n}{n}\right)^n \\ &\le \left(\sin\left(\frac{\psi_1 + \cdots + \psi_n}{n}\right)\right)^n = (\sin\alpha)^n.\end{aligned}$$

Observe that the above proof yields also the case of equality in the Lemma. In case of equality, $\sphericalangle PA_kA_{k+1} \geq \alpha$ for every k and $N_k = A_{k+1}$ is also possible but still $0 < \psi_k < \pi$. Then, in inequality (3), equality is also possible, and in fact equality is the only possibility as we showed. Thus $\psi_1 = \cdots = \psi_n$ which means $\psi_k = \alpha$ for every k, therefore, $A_1A_2 \dots A_n$ is a regular n-gon.

Many birds with one stone

We now give a collection of problems from the past few decades to illustrate the variety of ways in which the above lemma has appeared without the names of Dmitriev and Dynkin being mentioned (which might be due to the inaccessibility of their paper).

Brocard angle. For $n = 3$, the Lemma implies that the Brocard angle of a triangle is at most $\pi/6$ and equality holds provided that the triangle is equilateral. Moreover, if a convex n-gon admits a Brocard point, i.e., a point P such that the angles PA_kA_{k+1} are equal, then the corresponding Brocard angle is at most $\pi/2 - \pi/n$ with equality in the case of a regular n-gon. We note that there exist n-gons that do not have Brocard points, see [1] for such an example.

Contest problems. The Lemma for the particular case $n = 3$ was a problem of the International Mathematical Olympiad in 1991, see [11, p. 23]. This book contains three different solutions to the problem, one of them based on the maximal value of the Brocard angle in a triangle (likely to be well known for students preparing for Olympiads). The approach of applying the inequality of arithmetic and geometric means and Jensen's inequality comes also from there. This idea appears also in the research report [5] where the author provides an elegant proof of the maximal value of the Brocard angle in a way very similar to the above presented applied to the particular case $n = 3$.

For $n = 4$, the Lemma appeared as a problem of a national contest in India in 1991, see [2, p. 55]. It is interesting that the two solutions given therein are rather algebraic and trigonometric.

Monthly problems. The case of equality in the Lemma appeared two times in the Problem Section of the American Mathematical Monthly in the years 2000–2001. Problem 10824 (see [10]) asked to show that if there is a point P in a triangle such that the angles $\sphericalangle PAB = \sphericalangle PBC = \sphericalangle PCA = 30°$, then the triangle is equilateral; Problem 10904 claimed analogous statement for $n = 4$ (see [6]), moreover, it also raised the question of possible generalizations to n-gons. This latter subproblem was marked with an $*$ meaning that no solutions were then available, but as we see, there did already exist a published solution. The Monthly published a solution due to A. Nijenhuis who formulated essentially the lemma of Dmitriev and Dynkin, with a proof very similar to theirs (he used the method of Lagrange's multipliers to find the maximum of a product of trigonometric functions).

Historical remarks

The aim of the 1945 paper of Dmitriev and Dynkin was to determine the location of eigenvalues of stochastic matrices in the complex plane, i.e., matrices with nonnegative entries and with each row summing to 1. This question was initiated by Andrey Kolmogorov (1903–1987) in his seminar on Markov chains at Moscow State University. The problem was partially answered independently by Dmitriev and Dynkin who were students that time at the university. They published two papers on the subject in the years 1945–46 in which they transformed the linear algebra question to a geometrical one. Their first paper contained the Lemma of the present paper which is a nice example of a bridge between two distant concepts of mathematics.

It is remarkable how brilliantly talented the two young men were, Dmitriev enrolled the university at the age of 14, Dynkin at the age of 16. Later both became prominent scientists. The book [9, p. 29] recalls a funny episode when Kolmogorov was asked for advice on the matter of use computers in the physics institue. Kolmogorov replied: “Why do you need those computers? You have Kolya [Nikolai] Dmitriev, don’t you?” We refer to [12] for a brief biography of Dmitriev and to [4], a lecture recorded at Cornell University, where Professor Dynkin recounts his seventy years in mathematics, including the above mentioned problem.

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