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Picard's weighty proof of Chebyshev's sum inequality

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"Mathematics is a part of physics. Physics is an experimental science, a part of natural science. Mathematics is the part of physics where experiments are cheap."—declared the eminent Russian mathematician, Vladimir Igorevich Arnold (1937–2010) in an address on teaching mathematics [**3**]. Although Arnold's words might sound a bit presumptuous, it is common that behind a seemingly pure mathematical concept, quite natural physical principles lie. For example, George Pólya (1887–1985) devoted a whole chapter to physical mathematics in his book on plausible reasoning [**10**]. Further, some years ago Mark Levi in [**7**] revealed dozens of surprising links between mathematics and physics, including an electrical proof of the inequality of arithmetic and harmonic means, which also appeared recently in THIS MAGAZINE [**12**].

In this note, we continue along the above philosophy and focus attention on an historical and interesting physical demonstration—which dates back to the French mathematician, Émile Picard (1856–1941)—of the following familiar algebraic inequality for real numbers often referred to as Chebyshev's sum inequality (or Chebyshev's order inequality in [**11**]).

Theorem 1 (Chebyshev's sum inequality)**.** *If $u_1 \le u_2 \le \cdots \le u_n$ and $v_1 \le v_2 \le \cdots \le v_n$ (or both sequences are decreasing), then*

$$(u_1 + u_2 + \cdots + u_n)(v_1 + v_2 + \cdots + v_n) \le n(u_1v_1 + u_2v_2 + \cdots + u_nv_n). \quad (1)$$

As a special case when the two sequences are equal, Chebyshev's inequality becomes essentially the square of the inequality between the arithmetic mean and the root mean square.

Historical background

Chebyshev's sum inequality is named after Pafnuty Lvovich Chebyshev (1821–1894), one of the founding fathers of Russian mathematics. In a brief note [**4**] of 1882, he formulated the integral version of the above inequality in a rather general form and published its proof in the subsequent paper [**5**]. Chebyshev's general inequality implies, as a special case, that if the functions $u, v\colon [0,1] \to \mathbb{R}$ are increasing (or simultaneously decreasing), then

$$\left(\int_0^1 u\,dx\right) \cdot \left(\int_0^1 v\,dx\right) \le \int_0^1 uv\,dx. \quad (2)$$

This integral inequality was communicated by Chebyshev to the French mathematician, Charles Hermite (1822–1901) who then included it, with the extra assumption that u, v are nonnegative and strictly monotone, in the lecture notes [**1**, pp. 48–49.] written to his analysis course taught at the Sorbonne in the second semester of 1881–82. Hermite gave acknowledgment to Chebyshev, and then presented a proof due to

Picard. Picard's reasoning started with the reduction of the integral inequality to the discrete one, which then was placed into a physical setting based on the notion of the center of gravity. Although this proof seems lesser-known, it is one of the many spectacular encounters between mathematics and physics.

We first briefly recall the very intuitive concepts of torque (or moment) and center of gravity (or center of mass in other terminology) and formulate the physical principles on which Picard relied. After reproducing Picard's arguments, we shall also present a mechanical interpretation of the classical proof of Chebyshev's inequality based on the rearrangement inequality.

The center of gravity in the center of attention

Suppose there are point particles with positive masses $m_1, \dots, m_n$ located at coordinates $x_1, \dots, x_n$, respectively, on the real axis which we now consider as a weightless horizontal rod (see Figure 1).

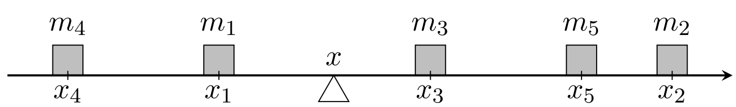

Figure 1

If the system is supported from below (or suspended) at a pivot point, then the downward gravitational pull on each mass results in a clockwise or counterclockwise rotation around the point. This turning effect is the so-called torque or moment, the concept of which was already used in mathematics by Archimedes of Syracuse when he calculated areas and volumes of various shapes with his ingenious method (see [**2**]). The magnitude of the torque equals the product of the weight of the particle (that is its mass times the gravitational acceleration g) and the lever arm (that is the distance between the particle and the pivot). Let us now work with signed torques: we assign a plus or minus sign to each torque depending on whether its rotational effect is counterclockwise or clockwise, respectively. Equivalently, the lever arm is considered as the signed distance between the particle and the pivot: it is negative when the particle is on the right side of the pivot. With this in mind, the torque about some pivot point x due to the particle with mass m_i is $m_i g(x - x_i)$. For simplicity, we can neglect the constant factor g, thus the total torque of the particles with respect to the point x becomes the sum $m_1(x - x_1) + m_2(x - x_2) + \cdots + m_n(x - x_n)$.

The center of gravity of a system of particles is the pivot point where the system is completely balanced when supported from below: the individual turning effects cancel each other and the total torque becomes zero. Thus, if the particle with mass m_i is located at x_i for all i and x_{cg} denotes the center of gravity, the condition of equilibrium takes the form

$$m_1(x_{\text{cg}} - x_1) + m_2(x_{\text{cg}} - x_2) + \cdots + m_n(x_{\text{cg}} - x_n) = 0$$

or equivalently

$$(m_1 + m_2 + \cdots + m_n)x_{\text{cg}} = m_1x_1 + m_2x_2 + \cdots + m_nx_n. \tag{3}$$

Therefore the center of gravity of the system becomes

$$x_{\text{cg}} = \frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{m_1 + m_2 + \cdots + m_n}. \tag{4}$$

In other words, the coordinate of the center of gravity is the weighted average of the coordinates of the masses. Consequently, if all the masses are equal, then the center of gravity is located at the arithmetic mean of the coordinates regardless of the particular value of the common mass.

Clearly, if a pivot point is located on the same side of each mass, then all the torques about this point act in the same direction and thus equilibrium cannot be achieved. This yields an important principle.

Principle 1. The center of gravity lies between the leftmost and rightmost of the masses.

The relation (3) expresses the fact that total torque of the system about the origin is the same as if the total mass of the system were concentrated at the center of gravity. Since the origin can be chosen freely, we obtain that the total torque of a system of masses about any point is identical with the turning effect of the total mass located at the center of gravity of the system. Practically, this implies another fundamental principle.

Principle 2. The center of gravity of a system does not change when a subsystem of its masses is replaced by the total mass of the subsystem concentrated at the center of gravity of the subsystem.

We still need a third intuitive observation. By removing the leftmost mass from the system, that is the one with the smallest coordinate, the clockwise torques overcome the counterclockwise torques about the original center of gravity, therefore the new center of gravity should be located on the right side of the original one.

Principle 3. By removing the leftmost/rightmost mass from a system, the center of gravity shifts to the right/left.

It is an easy exercise to derive all the above principles in a rigorous mathematical way by using the particular formula (4) for the center of gravity. Now we are ready to present Picard's arguments.

Picard's proof rephrased

By following Hermite and Picard, we first add the extra assumption to Chebyshev's inequality that the sequences are positive and strictly monotone. For aesthetic reasons, we rephrase Picard's proof in the case when the two sequences are simultaneously increasing. Let us start by rewriting inequality (1) in the more suggestive form

$$\frac{u_1+u_2+\cdots+u_n}{n} \le \frac{u_1v_1+u_2v_2+\cdots+u_nv_n}{v_1+v_2+\cdots+v_n}. \tag{5}$$

We now consider the points $A_1, A_2, \ldots, A_{n-1}, A_n$ on the real axis with coordinates $u_1 \le u_2 \le \cdots \le u_{n-1} \le u_n$ and concentrate mass v_i at the point A_i. Then the system's center of gravity x_{cg} becomes the right-hand side of (5). In view of the telescoping sum

$$v_k = v_1 + (v_2 - v_1) + (v_3 - v_2) + \cdots + (v_{k-1} - v_{k-2}) + (v_k - v_{k-1}),$$

our system can be decomposed into some subsystems as follows. In the first subsystem we concentrate mass v_1 at each point $A_1, A_2, \ldots, A_n$; in the second subsystem mass $v_2 - v_1$ at each point $A_2, A_3, \ldots A_n$ and so forth; in the nth system mass $v_n - v_{n-1}$

is concentrated at point A_n. For simplicity, let us denote the center of gravity of the ith subsystem by $x_{\text{cg}}^{(i)}$. Observe that $x_{\text{cg}}^{(1)}$ is exactly the left-hand side of (5). Now, by Principle 2, the center of gravity x_{cg} of the whole system does not change if we replace each subsystem by their total mass concentrated at $x_{\text{cg}}^{(i)}$. Therefore, Principle 1 implies that x_{cg} is located between the two extremes of the coordinates $x_{\text{cg}}^{(i)}$ $(i = 1, \ldots, n)$. We claim that $x_{\text{cg}}^{(1)} \leq x_{\text{cg}}^{(2)} \leq \cdots \leq x_{\text{cg}}^{(n)}$, thus $x_{\text{cg}}^{(1)} \leq x_{\text{cg}}$, being exactly inequality (5). The relation $x_{\text{cg}}^{(k)} \leq x_{\text{cg}}^{(\ell)}$ for $k < \ell$ follows by Principle 3, since a removal of the leftmost points $A_k, \ldots, A_{\ell-1}$ from the kth subsystem yields a system of equal masses concentrated at $A_\ell, \ldots, A_n$ which has the same center of gravity as the ℓth subsystem.

Looking back and ahead

It is readily seen from the proof that for strict monotone sequences the inequality is strict as well (in fact, this was proved by Picard). However, the proof applies also if we forget about the strict monotonicity condition. In case $v_k - v_{k-1} = 0$ for some (but not all) k, the omission of these subsystems will not affect the rest of the argument (and for constant (v_i) equality is evident). It is likewise admissible that some of the points A_i coincide.

Furthermore, the positivity assumption added by Hermite and Picard is also not a proper restriction. If the sequence (u_i) contains some nonnegative terms but (v_i) is positive, then we can choose some real number d such that $u_i + d > 0$ for all i (which is physically a translation of the origin). Hence

$$\big((u_1 + d) + \cdots + (u_n + d)\big)(v_1 + \cdots + v_n) \leq n\big((u_1 + d)v_1 + \cdots + (u_n + d)v_n\big)$$

from where d vanishes and inequality (1) follows. When both sequences contain nonnegative terms, the same substitution can be applied successively.

If $u, v\colon [0,1] \to \mathbb{R}$ are simultaneously increasing (or decreasing) functions, then by taking the partition $x_i = i/n$ $(i = 0, \ldots, n)$ of the interval $[0,1]$ and letting $u_i = u(x_i)$, $v_i = v(x_i)$, Chebyshev's sum inequality implies

$$\frac{1}{n}\sum_{i=1}^{n} u\left(\frac{i}{n}\right) \cdot \frac{1}{n}\sum_{i=1}^{n} v\left(\frac{i}{n}\right) \leq \frac{1}{n}\sum_{i=1}^{n} (uv)\left(\frac{i}{n}\right).$$

From here we obtain inequality (2) as $n \to \infty$.

A moment on rearrangement

Motivated by Picard's arguments, we now present a simple mechanical interpretation of the usual proof of Chebyshev's inequality carried out via the rearrangement inequality. To this end, assume that $x_1 \leq x_2 \leq \cdots \leq x_n$, $m_1 \leq m_2 \leq \cdots \leq m_n$ and let us allocate to each coordinate x_i exactly one particle so as to maximize the total torque of the system about the origin. Intuitively, the particle with the largest mass m_n should be associated with the largest lever arm x_n; the particle with the second largest mass to the second largest lever arm and so forth. Indeed, suppose we have a different arrangement in which j is the maximal index for which particle with mass m_j is not located at coordinate x_j but at some x_k where $k < j$. Then some particle with mass m_ℓ is located at x_j where $\ell < j$ (see Figure 2). Swapping the particles with masses m_j and m_ℓ (alone) the total torque about the origin does not decrease since it changes

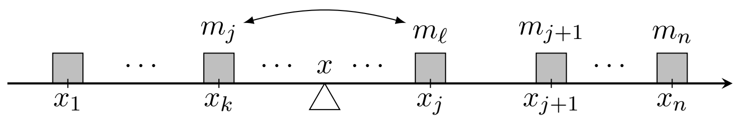

Figure 2

by

$$(m_\ell x_k + m_j x_j) - (m_j x_k + m_\ell x_j) = (m_j - m_\ell)(x_j - x_k) \geq 0.$$

In the new arrangement mass m_j is placed at x_j so by repeating this procedure we obtain that the masses are in increasing order and meanwhile the torque has not been decreased.

Note that the positivity of the masses is not essential in the previous argument. Otherwise we can add some mass d to each m_i to guarantee $m_i + d > 0$. Then in any arrangement the total torque increases by the amount $d(x_1 + x_2 + \cdots + x_n)$ which does not affect the order between the total torques. We have obtained the well-known rearrangement inequality.

Theorem 2 (Rearrangement inequality)**.** *Assume that* $u_1 \leq u_2 \leq \cdots \leq u_n$ *and* $v_1 \leq v_2 \leq \cdots \leq v_n$. *If* $\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n$ *is any permutation of the numbers* $v_1, v_2, \ldots, v_n$, *then*

$$u_1\tilde{v}_1 + u_2\tilde{v}_2 + \cdots + u_n\tilde{v}_n \leq u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Chebyshev's inequality might be interpreted similarly. We consider n^2 particles: n particles with mass v_i for all i where $v_1 \leq v_2 \leq \cdots \leq v_n$. Let us place at each coordinate $u_1 \leq u_2 \leq \cdots \leq u_n$ exactly n particles in such a way that the total torque is maximal. The same argument applies as before: the maximum is attained when all the n particles with mass v_i are assigned to u_i for all i. Then the total torque becomes the right-hand side of (1) while the left-hand side of (1) is the total torque of the system when the total mass located at u_i is $v_1 + v_2 + \cdots + v_n$ for all i. We have now obtained a physical explanation of both the rearrangement inequality and Chebyshev's inequality in terms of torque.

History goes on

The third edition of Hermite's lecture notes appeared in 1887 contained neither Picard's proof nor the discrete inequality (1). Only the integral version was demonstrated based on the nice and easy to check integral identity

$$\begin{aligned}\frac{1}{2}\int_a^b\int_a^b &\left[\big(f(x)-f(y)\big)\big(g(x)-g(y)\big)\right]\,dxdy\\ &= (b-a)\int_a^b f(x)g(x)\,dx - \left(\int_a^b f(x)\,dx\right)\cdot\left(\int_a^b g(x)\,dx\right).\end{aligned}$$

Although, Hermite ascribed this identity to the Hungarian born American mathematician Fabian Franklin (1853–1939) who used it in a paper of 1885, it was already established in 1883 by the Russian mathematician Konstantin Alekseevich Andreev

(1848–1921), that time professor at the University of Kharkov. The discrete analogue of the identity has the form

$$\frac{1}{n}\sum_{i=1}^{n} x_i y_i = \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) + \frac{1}{n^2}\sum_{i<j}(x_i - x_j)(y_i - y_j)$$

which is due to the Russian mathematician Aleksandr Nikolayevich Korkin (1837–1908), a former student of Chebyshev, who communicated it to Hermite in a letter of 1883. Nowadays, the usual proof of Chebyshev's inequality is based on the above algebraic identity or the rearrangement inequality.

We stop here in the journey, for more details we refer to the monograph [**9**, Chap. IX] where the complete history and evolution of Chebyshev's inequality and its generalizations is nicely collected together with accurate bibliographic details of the above cited papers as well. Finally, regarding many other applications of the center of gravity in mathematical demonstrations we highly recommend [**7**] for the interested reader.

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Summary. In his analysis course notes of 1882, Hermite included an algebraic inequality known today as Chebyshev's sum inequality. He presented a physical demonstration due to Picard which was based on the intuitive concept of the center of gravity. We first recall Picard's reasoning with some historical background and then, motivated by his idea, we provide a mechanical interpretation of the usual proof of Chebyshev's inequality carried out via the rearrangement inequality.

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