ON COMPLETE MONOTONICITY OF SOME FUNCTIONS RELATED TO MEANS

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ABSTRACT. We show the complete monotonicity of some functions related to the Stolarsky mean which was a problem of S.-X. Chen and F. Qi (2007) [15]. In the proof, the connection between operator monotone and completely monotonic functions is used.

1. INTRODUCTION

An infinitely differentiable function $f\colon I \to \mathbb{R}$ defined on an open interval I is said to be *completely monotonic* on I if $(-1)^n f^{(n)}(x) \geq 0$ for $x \in I$ and for every $n = 0, 1, 2 \ldots$. (It can be shown that for nonconstant functions the inequality is strict, see [9]). This definition is a continuous analogue of totally monotone sequences which was introduced by F. Hausdorff in 1921 (see [20]). Completely monotonic functions have applications in different areas of mathematics, for instance, in potential theory, probability theory, numerical and asymptotic analysis and combinatorics, see the references in [1]. The monographs [17, 20] provide a detailed discussion of the properties of completely monotonic functions. We refer to [4] for an excellent historical account on functions having derivatives of constant sign.

Recently, the complete monotonicity of numerous functions related to the gamma, polygamma, and other special functions was investigated by many authors, see [6, 7, 16, 18] and the references therein.

The present paper is motivated by the work [15] of S.-X. Chen and F. Qi. There the authors obtained the following result.

Theorem 1. *For all fixed $s, t \in \mathbb{R}$, the derivative of the function $x \mapsto L(x+s, x+t)$ is completely monotonic in $(-\min(s,t), +\infty)$ where L denotes the logarithmic mean:*

$$L(x,y) = \begin{cases} \frac{x-y}{\log x - \log y}, & \text{if } x \neq y, \\ x, & \text{if } x = y. \end{cases}$$

In [15], a function of which the derivative is completely monotonic is called *completely monotonic of first order* by the authors. Such functions have an extensive theory and they are called *Bernstein functions* in [17]. However, one may call them *completely monotone mappings* or *Laplace exponents* depending on the subject. Nevertheless, we adopt the following definition.

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Definition 1. An infinitely differentiable function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ is called a *Bernstein function* if $(-1)^{n-1}f^{(n)} \geq 0$ for all $n = 1, 2\ldots,$ i.e., if f' is completely monotonic.

Since the property of complete monotonicity is invariant under linear coordinate transformation, Theorem 1 says equivalently that $x \mapsto L(x+s, x)$ is a Bernstein function for every $s > 0$.

A possible generalization of the logarithmic mean was introduced by K. B. Stolarsky in [19]. The so-called Stolarsky (or extended) mean is defined as follows (see also [13]):

$$S_{p,q}(x,y) := \begin{cases} \left(\frac{q(x^p-y^p)}{p(x^q-y^q)}\right)^{\frac{1}{p-q}}, & \text{if } pq(p-q) \neq 0, x \neq y, \\ \exp\left(-\frac{1}{p} + \frac{x^p \log x - y^p \log y}{x^p - y^p}\right), & \text{if } p = q \neq 0, x \neq y, \\ \left(\frac{x^p-y^p}{p(\log x - \log y)}\right)^{\frac{1}{p}}, & \text{if } p \neq 0, q = 0, x \neq y, \\ \left(\frac{x^q-y^q}{q(\log x - \log y)}\right)^{\frac{1}{q}}, & \text{if } p = 0, q \neq 0, x \neq y, \\ \sqrt{xy}, & \text{if } p = q = 0, \\ x, & \text{if } x = y. \end{cases}$$

In recent years, the Stolarsky mean have been the subject of an intensive research, several of its properties have been investigated, such as monotonicity, comparison, logarithmic and Schur-convexity, see the references in [15]. However, there are no results on complete monotonicity. This motivated the authors of [15] in posing the following open problem: what can be said about the complete monotonicity property of the (derivative of the) function $x \mapsto S_{p,q}(x+s, x+t)$?

The main result of the present paper is the following.

Theorem 2. *The function $x \mapsto S_{p,q}(x+s, x)$ is a Bernstein function for all fixed $s > 0$ and $-1 \leq q \leq 1$, $-2 \leq p \leq 2$ or symmetrically $-1 \leq p \leq 1$, $-2 \leq q \leq 2$.*

For certain values of the parameters (p, q), the Stolarsky mean $S_{p,q}$ yields some well-known means. Namely, the power means (or Hölder means or binomial means)

$$S_{2p,p}(x,y) = H_p(x,y) = \begin{cases} \left(\frac{x^p+y^p}{2}\right)^{\frac{1}{p}}, & \text{if } p \neq 0, \\ \sqrt{xy}, & \text{if } p = 0, \end{cases}$$

the power difference means (see [3])

$$S_{p,p-1}(x,y) = K_p(x,y) = \begin{cases} \frac{p-1}{p} \cdot \frac{x^p-y^p}{x^{p-1}-y^{p-1}}, & \text{if } p \neq 0, 1, x \neq y, \\ \frac{x-y}{\log x - \log y}, & \text{if } p = 1, x \neq y, \\ xy\frac{\log x - \log y}{x-y}, & \text{if } p = 0, x \neq y, \\ x, & \text{if } x = y, \end{cases}$$

and the generalized logarithmic mean (see [5])

$$S_{p+1,1}(x,y) = L_p(x,y) = \begin{cases} \left(\frac{x^{p+1}-y^{p+1}}{(p+1)(x-y)}\right)^{\frac{1}{p}}, & \text{if } p \neq -1, 0, x \neq y, \\ \frac{x-y}{\log x - \log y}, & \text{if } p = -1, x \neq y, \\ \exp\left(-1 + \frac{x\log x - y \log y}{x-y}\right), & \text{if } p = 0, x \neq y, \\ x, & \text{if } x = y. \end{cases}$$

Corollary 3. *The function $x \mapsto M_{p,q}(x+s,x)$ is a Bernstein function for all fixed $s > 0$ where M may denote the following means:*

(i) H_p *for* $-1 \leq p \leq 1$;
(ii) K_p *for* $-1 \leq p \leq 2$;
(iii) L_p *for* $-3 \leq p \leq 1$.

In the proof of Theorem 2, we follow a different approach than that of [15]. Our argument is based on the fact that Bernstein functions have close relationship to operator monotone functions, namely, operator monotone functions are also Bernstein functions. We note that operator monotone functions play an important role in the theory of matrix means, see [12]. The operator monotonicity of some functions were investigated in [10, 8], however, the Stolarsky mean did not appear, nor the notion of complete monotonicity. One purpose of this paper is to draw the attention to the relation of the two notions.

In the next Section, we briefly summarize the results corresponding to Bernstein and operator monotone functions. For a detailed discussion, see the monographs [2, 17].

2. Complete monotonicity and operator monotonicity

Let $\mathbb{M}_n^+$ denote the space of $n \times n$ complex Hermitian positive-semidefinite (briefly positive) matrices with the usual ordering: $A \leq B$ means that $B - A$ is positive. Let f be a real function on an interval I. If D is a diagonal matrix with diagonal entries $\lambda_j \in I$ $(j = 1, \ldots, n)$, then $f(D) := \operatorname{diag}(f(\lambda_1), \ldots, f(\lambda_n))$. If A is an Hermitian matrix with eigenvalues in I then let $f(A) := Uf(D)U^*$ where $A = UDU^*$ with unitary matrix U and diagonal matrix D containing the eigenvalues of A.

Definition 2. A function $f\colon I \to \mathbb{R}^+$ is called *matrix monotone of order n* if $A \leq B$ implies $f(A) \leq f(B)$ for every $A, B \in \mathbb{M}_n^+$ having eigenvalues in the interval I. If f is matrix monotone of order n for all n then f is called *operator monotone* (on I).

As a trivial consequence of the definition, we obtain an important property of operator monotone functions.

Proposition 4. *The pointwise limit of operator monotone functions on I is again an operator monotone function on I.*

The theory of operator monotone functions began with the seminal paper [14] of K. Löwner. In that paper, he established the connection between operator monotone functions, Pick functions (see later) and the positivity of the matrix of divided differences. The following characterization is due to him.

Theorem 5 (Löwner). *A function $f\colon I \to \mathbb{R}^+$ is operator monotone if and only if it has a holomorphic continuation to a mapping of the open upper half-plane into the closed upper half-plane.*

In the literature, there are various names for holomorphic functions which map the open upper half-plane into the closed upper half-plane. Depending on the subject, one may call them *Herglotz*, *Pick* or *Nevanlinna* functions (or with some combination of these names), however, we shall use the term Pick function. We note that by the open mapping theorem, nonconstant Pick functions map the open upper half-plane into the open upper half-plane. The following integral characterization of Pick functions is known (see [2]).

Theorem 6. *Suppose that $F\colon \{\operatorname{Im} z > 0\} \to \{\operatorname{Im} z \geq 0\}$ is holomorphic. Then*

$$F(z) = \alpha + \beta z + \int_{-\infty}^{+\infty} \left(\frac{1}{\lambda - z} - \frac{\lambda}{\lambda^2 + 1} \right) d\mu(\lambda),$$

where $\alpha \in \mathbb{R}$, $\beta \geq 0$ and μ is a Borel-measure on $\mathbb{R}$ for which

$$\int_{\mathbb{R}} \frac{1}{\lambda^2 + 1} \, d\mu(\lambda) < \infty.$$

Conversely, a function having such form maps the open upper half-plane holomorphically into the closed upper half-plane.

Combining the characterization of Pick functions and Löwner's theorem one may deduce an integral representation of operator monotone functions (see [2]) which was first used by E. Heinz in [11].

Theorem 7. *A function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ is operator monotone if and only if it has the representation*

$$f(t) = a + bt + \int_0^{\infty} \left(\frac{\lambda}{\lambda^2 + 1} - \frac{1}{\lambda + t} \right) d\mu(\lambda)$$

with $a \in \mathbb{R}$, $b \geq 0$ and a Borel measure μ on $\mathbb{R}^+$ such that

$$\int_0^{\infty} \frac{d\mu(\lambda)}{\lambda^2 + 1} < \infty.$$

Clearly, for every fixed $\lambda > 0$, the function $t \mapsto -1/(\lambda + t)$ is completely monotonic in $\mathbb{R}^+$. Since this property is preserved under the integration (with respect to the parameter λ), we obtain the connection between operator monotone and completely monotone functions (see [2, 17]).

Corollary 8. *If the function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ is operator monotone then it is a Bernstein function (i.e., its derivative is completely monotonic).*

Remark 9. We note that the class of positive operator monotone functions coincides with a proper subset of Bernstein functions which is called complete Bernstein functions, see [17]. This fact suggests that Theorem 2 might not be sharp. Indeed, numerical experiments show that in case $p \cdot q < 0$ the complete monotonicity holds on a larger set than given in Theorem 2.

3. Proof of the main result

We show that the function $x \mapsto S_{p,q}(x+s,x)$ is operator monotone for every fixed $s>0$ and by Corollary 8 this implies that it is also a Bernstein function. In order to show operator monotonicity, we verify that the function $x \mapsto S_{p,q}(x+s,x)$ has a holomorphic continuation to a mapping of the upper half-plane such that

$$\arg(z+s) < \arg S_{p,q}(z+s,z) < \arg z \qquad (\operatorname{Im} z > 0). \tag{1}$$

We shall use an idea of Ando, see [10]. Below, the complex logarithm is defined the usual way: $\log z = \log|z| + \mathrm{i}\arg z$ where $0 \le \arg z < 2\pi$.

First, notice that the limiting cases $p=0$, $q=0$ and $p=q$ follow by passing to the limit and employing Proposition 4. Assume $0<q<p<2$ (hence $0<q\le 1$). By a simple calculation we find that

$$\frac{q((x+s)^p - x^p)}{p((x+s)^q - x^q)} = \int_0^1 (\lambda(x+s)^q + (1-\lambda)x^q)^{\frac{p}{q}-1}\, d\lambda.$$

Clearly, if $s>0$, $0<\lambda<1$ and $\operatorname{Im} z>0$, then

$$q\arg(z+s) < \arg\left(\lambda(z+s)^q + (1-\lambda)z^q\right) < q\arg z,$$

hence

$$(p-q)\arg(z+s) < \arg\left((\lambda(z+s)^q + (1-\lambda)z^q)^{\frac{p}{q}-1}\right) < (p-q)\arg z.$$

After integration with respect to λ we obtain

$$0 < (p-q)\arg(z+s) < \arg\left(\frac{q((z+s)^p - z^p)}{p((z+s)^q - z^q)}\right) < (p-q)\arg z < 2\pi, \tag{2}$$

which yields (1). The case $0<p<q<2$ follows by symmetry.

Let $p<q<0$. Then

$$\left(\frac{q((x+s)^p - x^p)}{p((x+s)^q - x^q)}\right)^{\frac{1}{p-q}} = \left((x(x+s))^{|p|-|q|}\frac{|p|\left((x+s)^{|q|} - x^{|q|}\right)}{|q|\left((x+s)^{|p|} - x^{|p|}\right)}\right)^{\frac{1}{|p|-|q|}}.$$

By (2), for $\operatorname{Im} z>0$ we have

$$(|q|-|p|)\arg z < \arg\left(\frac{|p|\left((z+s)^{|q|} - z^{|q|}\right)}{|q|\left((z+s)^{|p|} - z^{|p|}\right)}\right) < (|q|-|p|)\arg(z+s),$$

and obviously

$$\arg\left(z(z+s)\right) = \arg z + \arg(z+s), \tag{3}$$

whence inequality (1) follows immediately. The case $q<p<0$ is obtained again due to symmetry.

Now, let $q<0<|q|<p$. Then

$$\left(\frac{q((x+s)^p - x^p)}{p((x+s)^q - x^q)}\right)^{\frac{1}{p-q}} = (x(x+s))^{\frac{|q|}{p+|q|}}\left(\frac{|q|\left((x+s)^p - x^p\right)}{p\left((x+s)^{|q|} - x^{|q|}\right)}\right)^{\frac{1}{p+|q|}}.$$

By (2), for $\operatorname{Im} z>0$,

$$(p-|q|)\arg(z+s) < \arg\left(\frac{|q|\left((z+s)^p - z^p\right)}{p\left((z+s)^{|q|} - z^{|q|}\right)}\right) < (p-|q|)\arg z,$$

which, together with (3) implies (1).

Finally, let $q < 0 < p < |q|$. Then

$$\left(\frac{q((x+s)^p - x^p)}{p((x+s)^q - x^q)}\right)^{\frac{1}{p-q}} = (x(x+s))^{\frac{|q|}{p+|q|}} \left[\left(\frac{p\left((x+s)^{|q|} - x^{|q|}\right)}{|q|\left((x+s)^p - x^p\right)}\right)^{\frac{1}{p+|q|}}\right]^{-1},$$

and one may use (2) and (3) to obtain (1).

Therefore, we have shown that the function $x \mapsto S_{p,q}(x+s,x)$ has a holomorphic continuation to a mapping of the upper half-plane into itself and thus it is operator monotone and so a Bernstein function.

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